1. Transfer function

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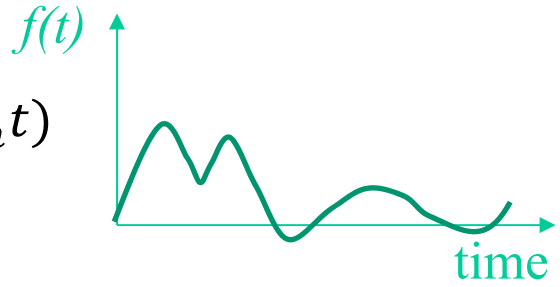
1.1 Laplace transform and Z- transform

1.1 Laplace transform

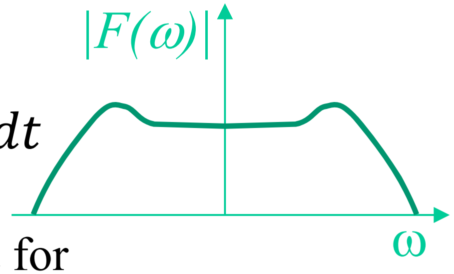
Time domain $f(t) = \sum_{n=0}^{\infty} (b_n \cos\omega_n t + a_n \sin\omega_n t)$



⇩

Fourier transform

Frequency domain $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$



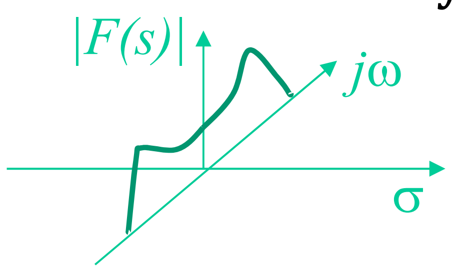
↕ Equivalent for periodic waveform signals

Laplace transform

s domain

$s = \sigma + j\omega$

$$F(s) = \int_{-\infty}^{\infty} f(t) u(t) e^{-(\sigma + j\omega)t} dt = \int_{0}^{\infty} f(t) e^{-(\sigma + j\omega)t} dt$$



Z-transform of discrete time signal

$$f_d(t) = \sum_{n=-\infty}^{\infty} f_n \delta(t - nT_S)$$

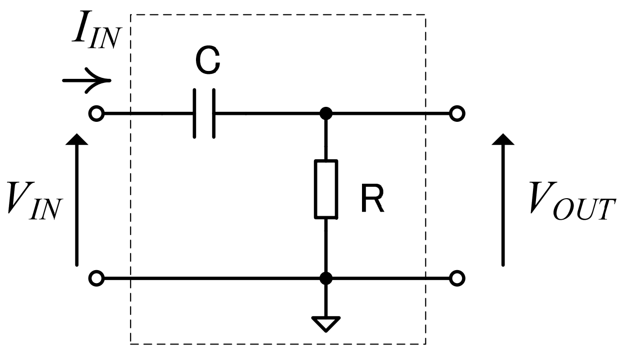
T_S: Sampling Period

$$F_d(s) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f_n \delta(t - nT_S)\, u(t) e^{-st} dt = \sum_{n=0}^{\infty} f_n e^{-snT_S}$$

$$F_d(z) = \sum_{n=0}^{\infty} f_n z^{-n}$$

$z = e^{sT_S}$

Example of transfer function of RC circuit



$$\begin{cases} v_{in}(t) = \dfrac{1}{C}\displaystyle\int_0^{\infty} i_{in}(t)dt + Ri_{in}(t) \\ v_{out}(t) = Ri_{in}(t) \end{cases}$$

$$\Big\downarrow \mathcal{L}$$

$$\begin{cases} V_{IN}(s) = \frac{1}{C}\left(\frac{I_{IN}(s)}{s} + \frac{q(t=0)}{s}\right) + RI_{IN}(s) = \left(\frac{1}{sC} + R\right) I_{IN}(s) \\ V_{OUT}(s) = RI_{IN}(s) \end{cases}$$

, q(t = 0) = 0 とする

$$H(s) = \frac{V_{OUT}(s)}{V_{IN}(s)} = \frac{R}{\frac{1}{Cs} + R} = \frac{sRC}{1 + sRC} \xrightarrow{s = j\omega} H(\omega) = \frac{V_{OUT}(\omega)}{V_{IN}(\omega)} = \frac{j\omega RC}{1 + j\omega RC}$$

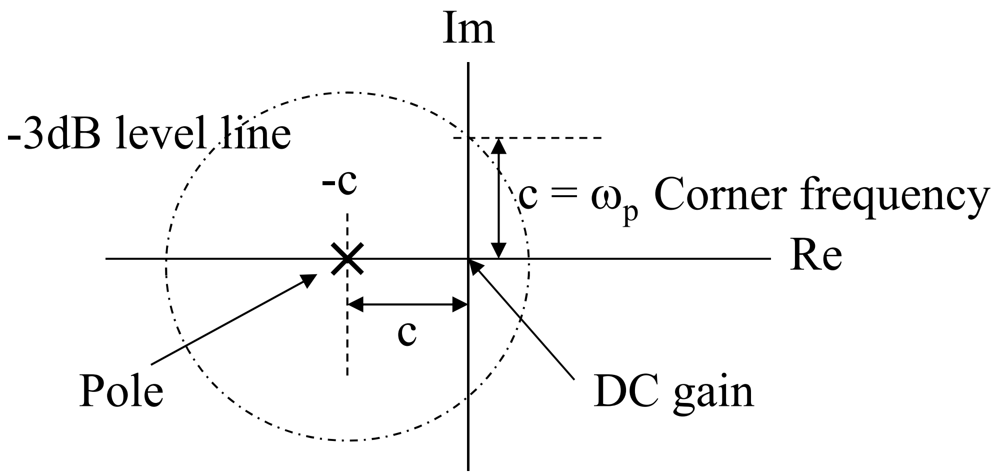
Knack: Deformation the transfer function into the form of (1±sX) or (1±$j\omega$X).

Bode diagram

$$H(s) = \frac{b}{s+c}$$

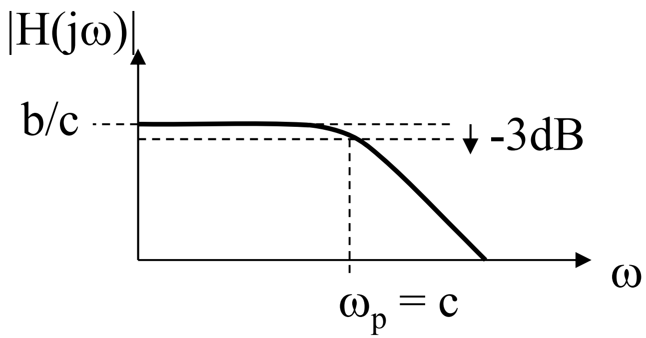
$$H(j\omega) = \frac{\frac{b}{c}}{1 + j\frac{\omega}{c}}$$

$$|H(j\omega)| = \frac{\frac{b}{c}}{\sqrt{1 + \frac{\omega^2}{c^2}}}$$



$$\begin{cases} \omega << c \rightarrow |H(j\omega)| = \frac{b}{c} \\ \omega = c \rightarrow |H(j\omega)| = \frac{1}{\sqrt{2}}\frac{b}{c} \\ \omega >> c \rightarrow |H(j\omega)| = \frac{b}{\omega} \end{cases}$$

$$\frac{1}{\sqrt{2}} = -3\text{dB}$$



Decibel (dB)

The vertical axis of Bode diagrams is plotted in the decibel scale (dB).The decibel indicates the absolute value ratio of the signal amplitude.

Decibel of voltage and current signal $dB = 20log_{10}\left|\frac{V2}{V1}\right| = 20log_{10}|H(\omega)|$

Decibel of signal power $dB = 10log_{10}\left|\frac{P2}{P1}\right|$

↑
Note

Note: dBm is not ratio, but the absolute value of the signal power in mW.

$$dBm = 10log_{10}P(mW)$$

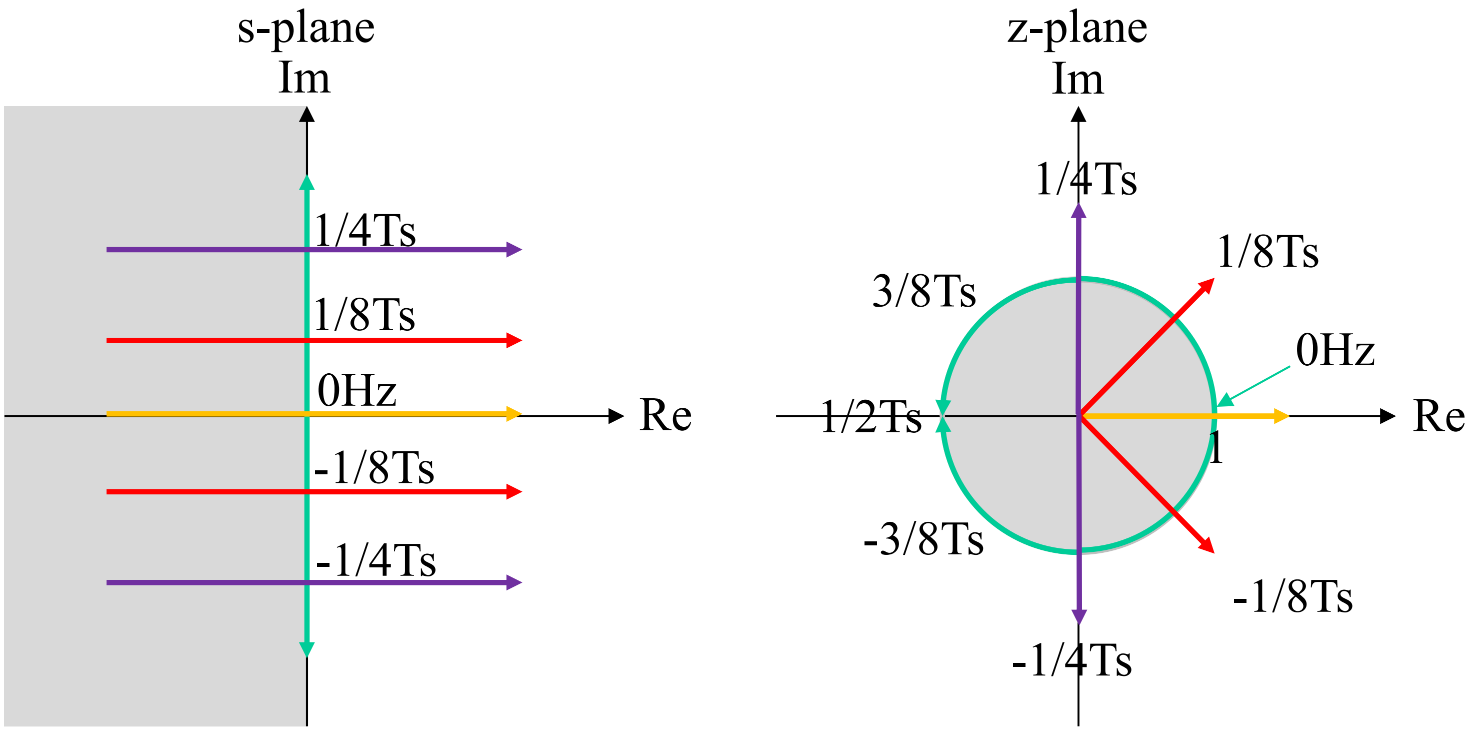
Check the typical value in Decibel

Calculate the typical value in Decibel.

0dB = ?
3dB = ?
-3dB = ?
6dB = ?
-6dB = ?
+20dB = ?
+40dB = ?
-20dB = ?
-40dB = ?
+20dB/Dec for the function of w = ?
-20dB/Dec for the function of w = ?
+40dB/Dec for the function of w = ?
-40dB/Dec for the function of w = ?

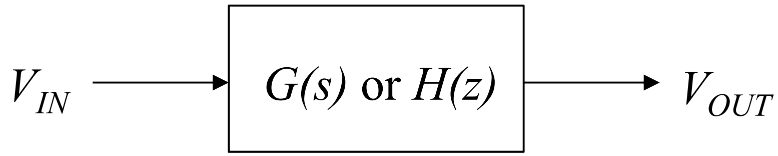
Correspondence relation of s-domain and z-domain

$$z = e^{(\sigma + j\omega)T_s}$$



1.2 Definition of transfer function

Definition of transfer function



- Transfer function of CT circuit

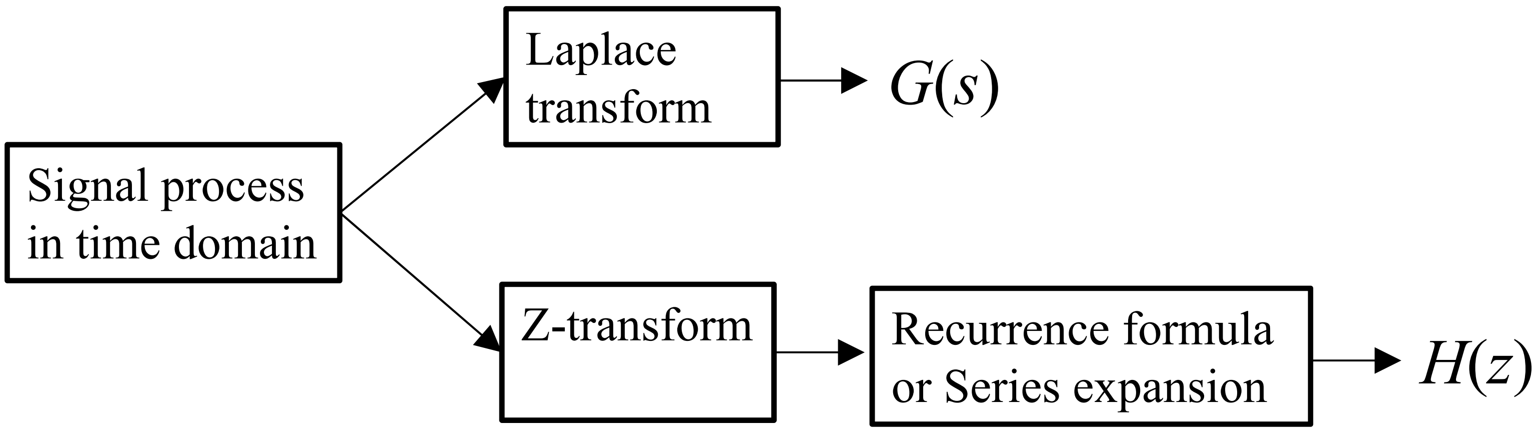
$$G(s) = \frac{V_{OUT}\ (s)}{V_{IN}\ (s)}$$

- Transfer function of DT circuit

$$H(z) = \frac{V_{OUT}\ (z)}{V_{IN}\ (z)}$$

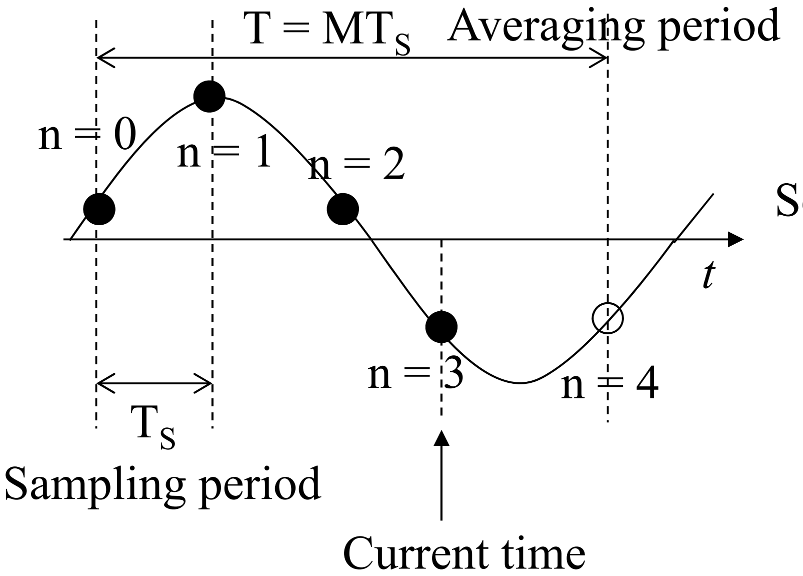
[NOTE] A transfer function can be defined in s-plane or z-plane.

Method of Deriving the transfer function



Example of moving average

In the case of M = 4



Signal process in time domain

$$y(t_n) = \frac{1}{T}\sum_{n=0}^{M-1} x(t_n)T_S = \frac{1}{M}\sum_{n=0}^{M-1} x(t_n)$$

$$= \frac{1}{M}\{x(t_0) + x(t_1) + x(t_2) + x(t_3)\}$$

$$\downarrow \mathcal{Z}$$

Series expansion of z variable

$$Y(z) = \frac{1}{M}\sum_{n=0}^{M-1} z^{-n}X(z)$$

$$= \frac{1}{M}\left(1 + z^{-1} + \cdots + z^{-(M-1)}\right)X(z)$$

$$= \frac{1}{M}\ (z^{-3} + z^{-2} + z^{-1} + z^{0})X(z)$$

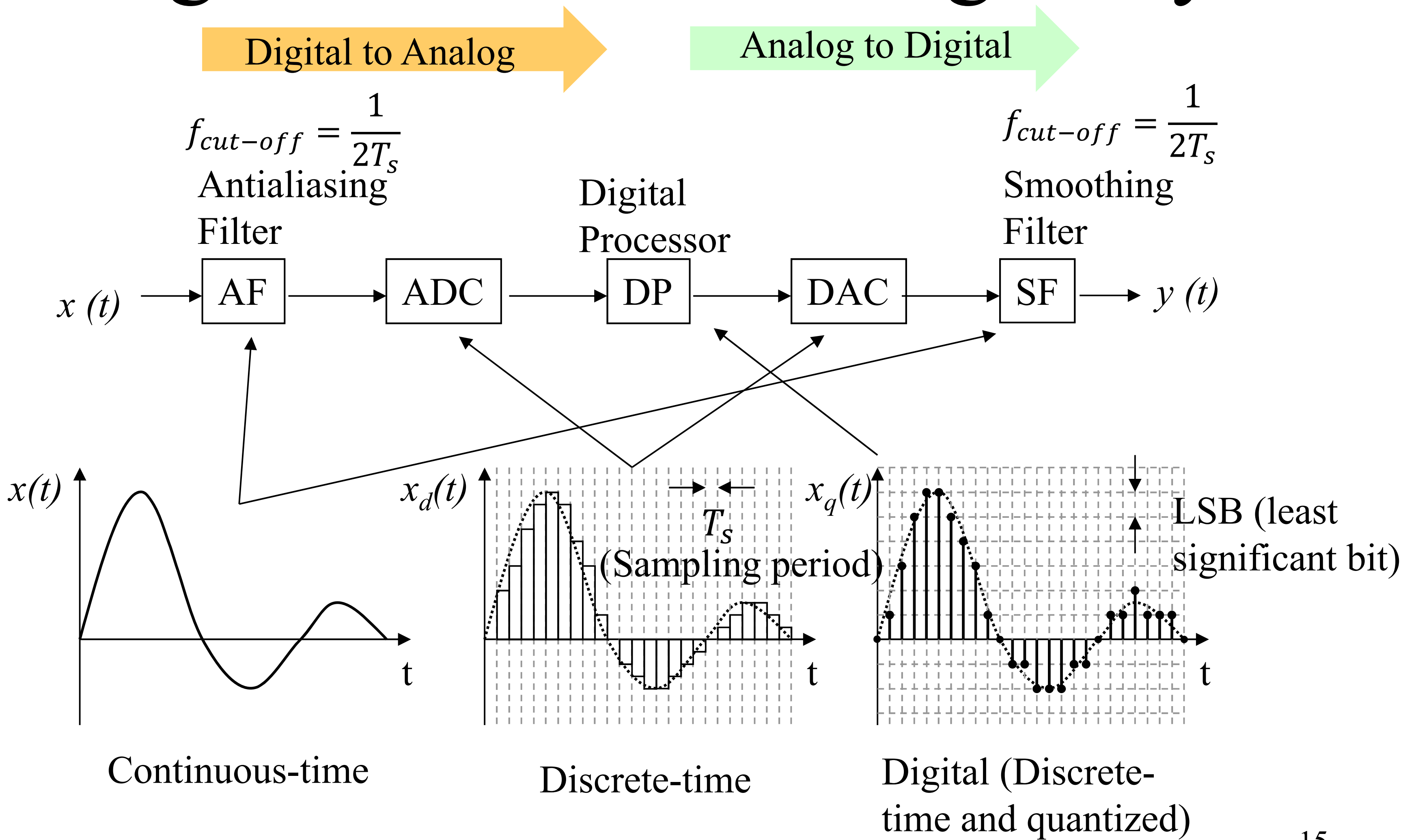
Transfer function

$$MY(z) = (z^{-1} + z^{-2} + \cdots + z^{-M})X(z) + (z^{0} - z^{-M})X(z) = z^{-1}MY(z) + (1 - z^{-M})X(z)$$

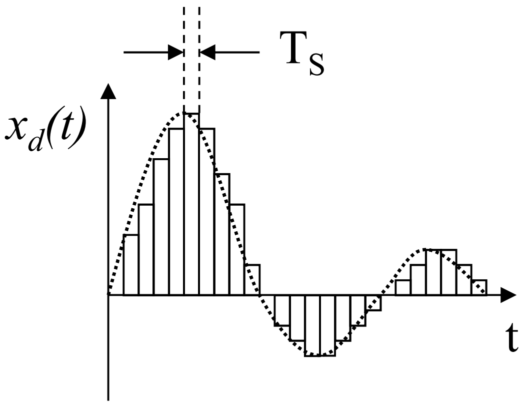
$$(1 - z^{-1})MY(z) = (1 - z^{-M})X(z) \longrightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1}{M}\frac{(1 - z^{-M})}{(1 - z^{-1})}$$

1.3 Waveforms in mixed-signal circuits

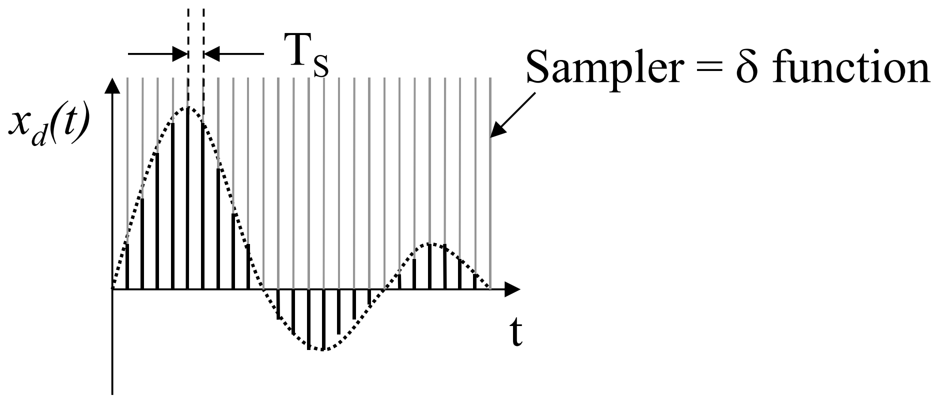
Configuration of mixed-signal system



Expressions of discrete-time signal



Step sampling
(Sample & Hold or S/H)



Impulse sampling
(Pulse Amplitude Modulation or PAM)

$$x_{du}(t)=\sum_n x(nT_S)\cdot\underbrace{\{u(t-nT_S)-u(t-(n+1)T_S)\}}$$

Step Sampler

$$x_d(t)=\sum_n x(t)\cdot\underbrace{\delta(t-nT_S)}$$

Impulse Sampler

NOTE: The discrete analog signal is practically obtained by S/H, but the signal can be handled similar to the impulse sequences.

Laplace transform and Z transform

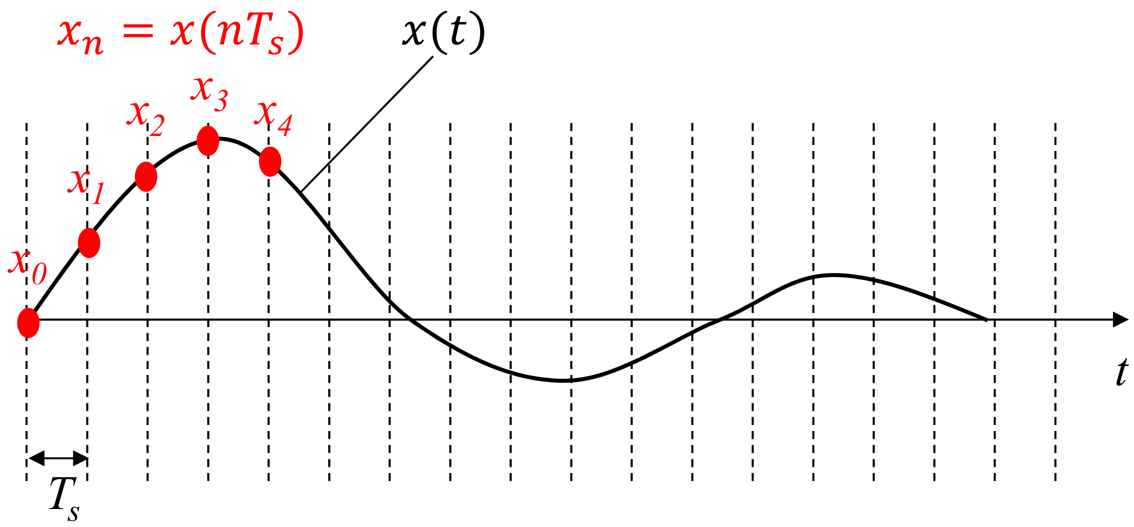
Continuous-time signal *x(t)* $\xrightarrow{\mathcal{L}}$ $X(s) = \int_0^{\infty} x(t)e^{-st}dt$

Discrete-time signal *x(t)* $\xrightarrow{\mathcal{Z}}$ $X(z) = \sum_{n=0}^{\infty} x_n z^{-n}$

$s = \sigma + j\omega$

↓

$z^{-1} = e^{-sT_s}$



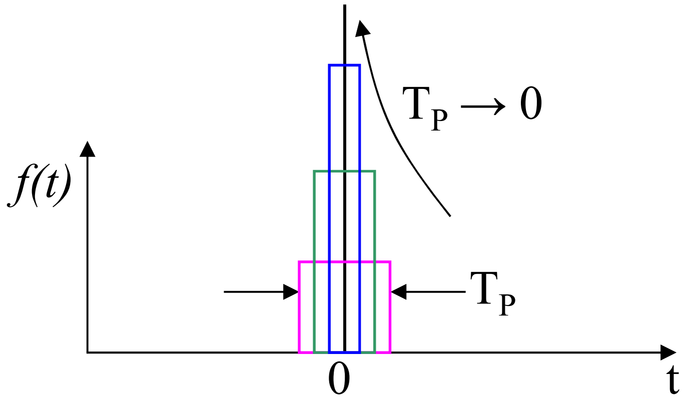
Characteristic of δ function

Step function *f*

$$f(t) = \frac{1}{T_P}\{u(t + \frac{T_P}{2}) - u(t - \frac{T_P}{2})\}$$

$$\int_{-\infty}^{\infty} f(t)dt = 1 \qquad (1)$$

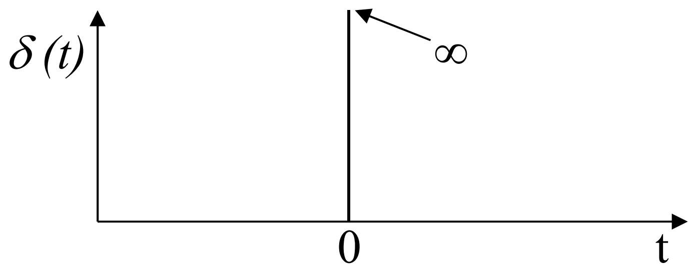
$$\delta(t) \equiv \lim_{T_P \to 0} f(t) \qquad (2)$$



Delta function δ

Important

$$\begin{cases} \int_{-\infty}^{\infty} \delta(t)dt = 1 & (1) \\ \delta(t) = 0 \quad (t \neq 0) & (2) \end{cases}$$



Z transform of discrete-time signal

PAM signal

$$x_d(t) = \sum_n x(t)\delta(t - nT_s)$$

⇩ Laplace transform

$$X_d(s) = \int_0^\infty x_d(t)e^{-sT_s}dt$$

$$= \int_0^\infty \sum_n x(t)\delta(t - nT_s)s^{-st}dt$$

$$= \sum_n \int_0^\infty x(t)\delta(t - nT_s)s^{-st}dt$$

$$= \sum_n x(nT_s)e^{-s(nT_s)}$$

$$= \sum_n x(nT_s)z^{-n}$$

$z^{-1} = e^{-sT_s}$

S/H signal

$$x_u(t) = \sum_n x(t)\{u(t - nT_s) - u(t - (n+1)T_s)\}$$

⇩ Laplace transform

$$X_u(s) = \int_0^\infty x_u(t)e^{-sT_s}dt$$

$$= \int_0^\infty \sum_n x(t)\{u(t - nT_s) - u(t - (n+1)T_s)\}e^{-st}dt$$

$$= \sum_n x(nT_s) \int_0^\infty \{u(t - nT_s) - u(t - (n+1)T_s)\}e^{-st}dt$$

$$= \sum_n x(nT_s) \frac{e^{-snT_s}(1 - e^{-sT_s})}{s}$$

$$= \underbrace{\frac{1 - e^{-sT_s}}{s}}_{\text{Transfer function of step sampler}} \underbrace{\sum_n x(nT_s)z^{-n}}_{\text{PAM signal}}$$

$z^{-1} = e^{-sT_s}$

Transfer function of step sampler

PAM signal

Z transform of PAM signal

Continuous-time signal $x(t)$ $\xrightarrow{\text{Laplace transform}}$ $X(s)$ $\xrightarrow{z = e^{sT_s}}$ $X(z)$

$\downarrow$ Impulse sampling

Discrete-time signal

$$x_d(t) = \sum_n x(nT_s)\delta(t - nT_s)$$

$\downarrow$ Z transform

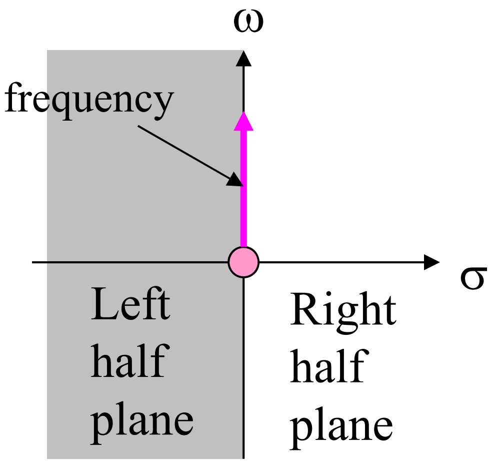
$$X_d(z) = \sum_n z_n z^{-n} \qquad (1)$$

A discrete-time signal can be easily transformed by using Eq.(1).
You do not need to calculate with Laplace transform.

s-plane and z-plane

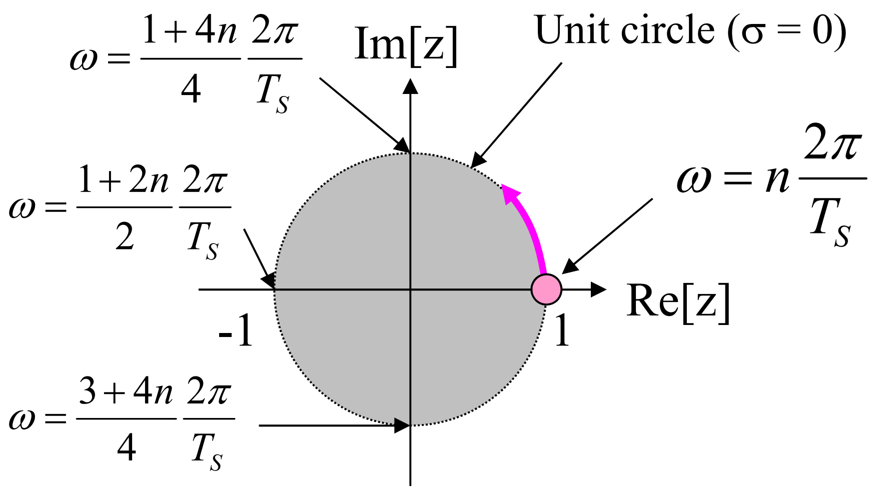
s-plane	Time domain	z-plane
$X_d(s) = \sum_{n} x(nT_s) e^{-nT_s}$ ←	$x_d(t) = \sum_{n=0}^{\infty} x(t)\delta(t - nT_s)$	→ $X_d(z) = \sum_{n=0}^{\infty} x_n z^{-n}$

s-plane



$$s = \sigma + j\omega$$

z-plane

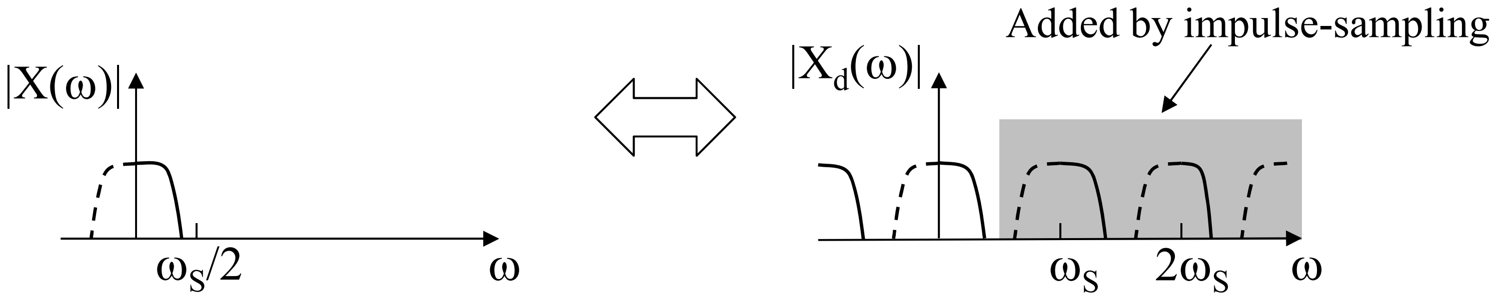


$$z = e^{sT_S} = e^{\sigma T_S}(\cos\omega T_S + j\sin\omega T_S)$$

Spectrum of PAM signal

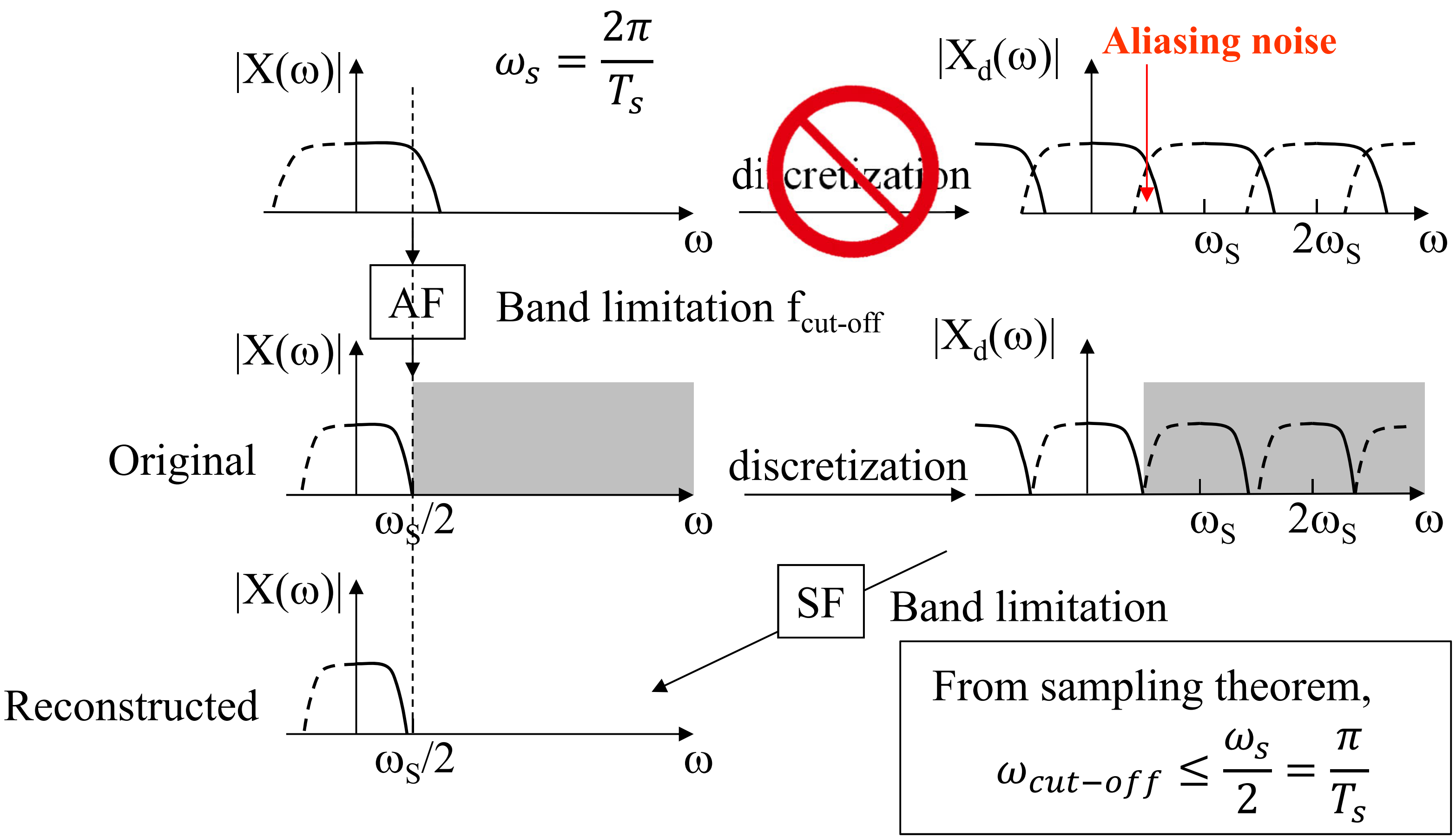
Spectrum of continuous-time signal

Spectrum of discrete-time signal



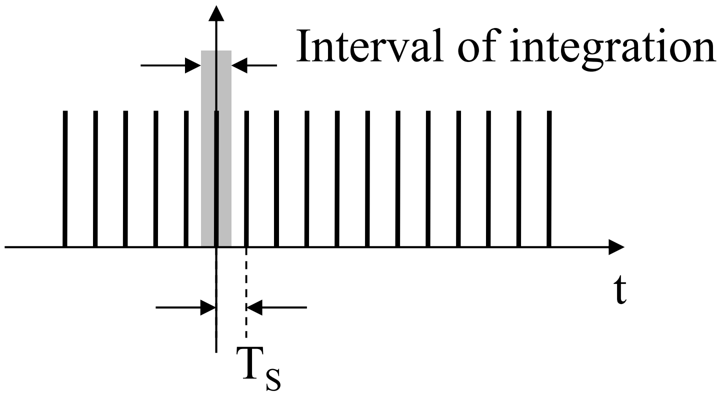
See Appendix 2(1) - 2(3).

Antialiasing and smoothing of signals



NOTE: AF and SF can be implemented in continuous-time circuits.

Appendix 2(1) Fourier series of impulse sampler



$$\delta_T(t) = \sum_n \delta(t - nT_S)$$

$$\delta_T(t) = \sum_n c_n e^{jn\frac{2\pi}{T_S}t}$$

$$c_n = \frac{1}{T_S}\int_{-\frac{T_S}{2}}^{\frac{T_S}{2}} \delta_T(t) \cdot e^{jn\frac{2\pi}{T_S}t} dt$$

$$= \frac{1}{T_S} e^{jn\frac{2\pi}{T_S}0} = \frac{1}{T_S}$$

$$\therefore \delta_T(t) = \frac{1}{T_S}\sum_n e^{jn\frac{2\pi}{T_S}t}$$

Appendix 2(2) Spectrum of PAM

$$x_d(t) = \sum_n x(t) \cdot \delta(t - nT_S) = x(t) \cdot \delta_T(t) = \frac{1}{T_S} \sum_n x(t) \cdot e^{jn\frac{2\pi}{T_S}t}$$

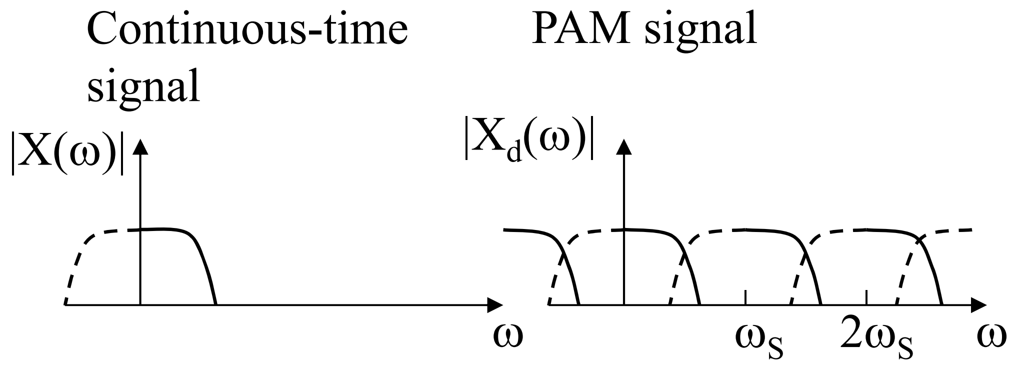
↓ Translation Theorem $\mathcal{L}[e^{-at} f(t)] = F(s+a)$

$$X_d(s) = \frac{1}{T_S} \sum_n X(s - jn\frac{2\pi}{T_S}) = \frac{1}{T_S} \sum_n X(s - jn\omega_S)$$

$$\omega_S = \frac{2\pi}{T_S}$$

Let's say, $s = j\omega$

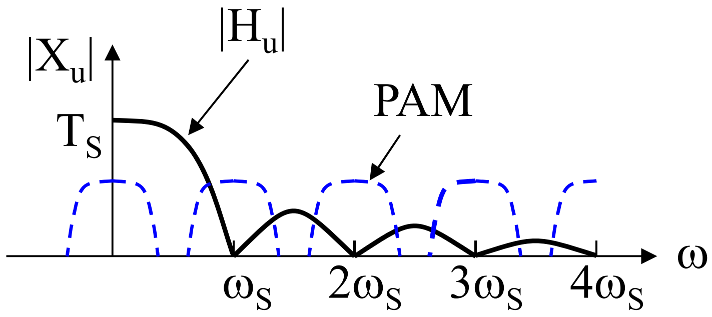
$$X_d(\omega) = \frac{1}{T_S} \sum_n X\{j(\omega - n\omega_S)\}$$



Appendix 2(3) Spectrum of S/H

Laplace transform of S/H signal (Slide 19)

$$X_u(s) = \underbrace{\boxed{\frac{1-e^{-sT_s}}{s}}}_{\text{Transfer function of step sampler } G_u(s)} \underbrace{\sum_n x(nT_s) z^{-n}}_{\text{PAM signal}}$$

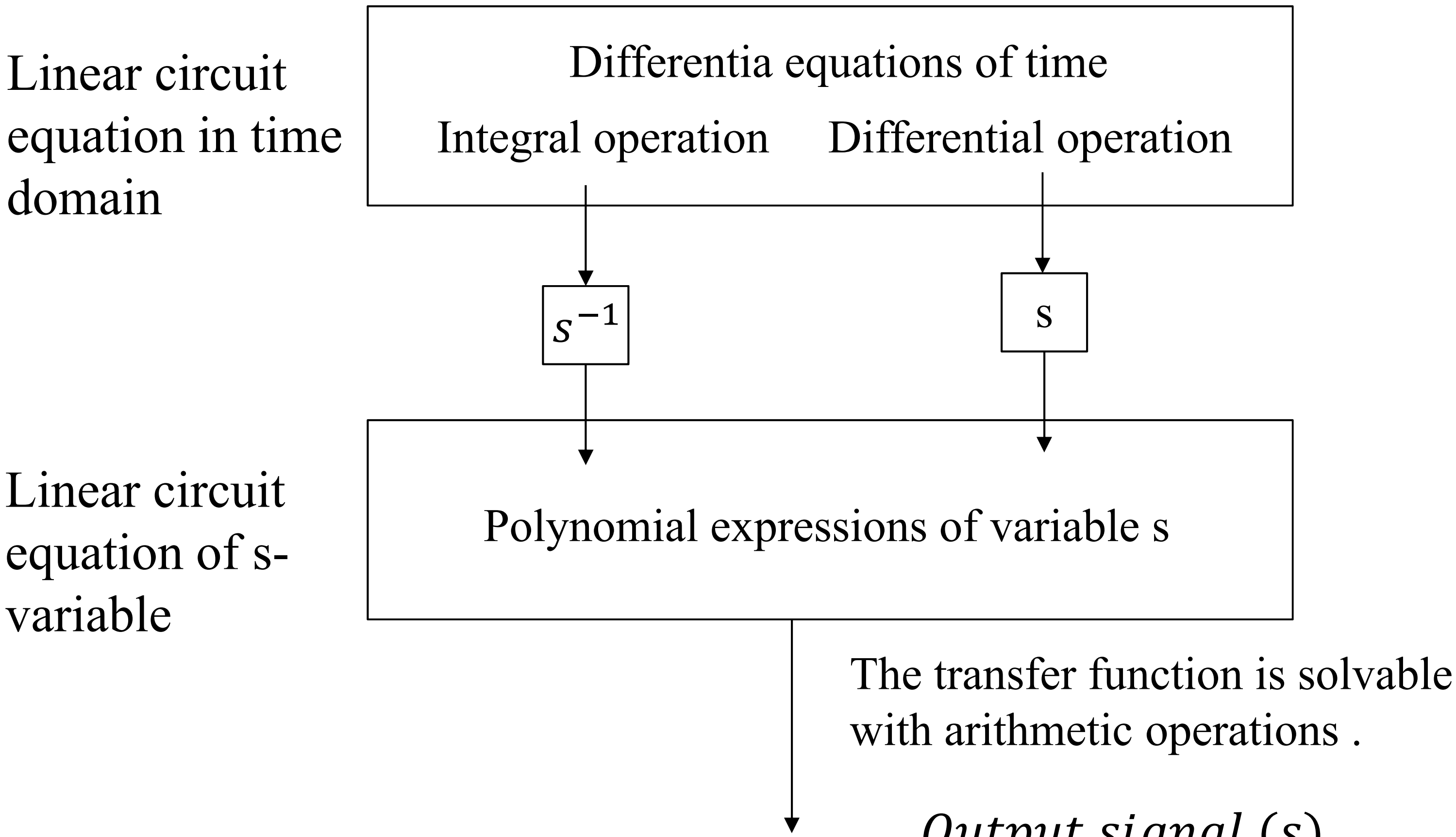


$$H_u(s) = \frac{1-e^{-sT_S}}{s} = e^{-\frac{sT_S}{2}} \frac{e^{\frac{sT_S}{2}} - e^{-\frac{sT_S}{2}}}{s} \xrightarrow{s=j\omega} e^{-j\frac{\omega T_S}{2}} \frac{e^{\frac{j\omega T_S}{2}} - e^{-\frac{j\omega T_S}{2}}}{j\omega} = e^{-j\frac{\omega T_S}{2}} T_S \frac{\sin\frac{\omega T_S}{2}}{\frac{\omega T_S}{2}} \equiv \underbrace{e^{-j\frac{\omega T_S}{2}}}_{\text{Phase}} \underbrace{T_S \cdot \mathrm{Sinc}(\frac{\omega T_S}{2})}_{\text{Amplitude}}$$

NOTE: The spectrum of S/H signal is deviated from the spectrum of PAM signal by the step sampler $H_u(s)$. Therefore, the smoothing filter after DAC must has the Sinc^{-1} characteristic.

1.4 Transfer function of continuous-time analog circuits

Integration and Differentiation



Transfer function $H(s) = \dfrac{Output\ signal\ (s)}{Input\ signal\ (s)}$

Definition of transfer function

- Transfer function
  - $s = \sigma + j\omega$: Transfer function
  - $s = j\omega$: Frequency transfer function

$$H(s) = \frac{Output\ signal\ (s)}{Input\ signal\ (s)}$$

- Pole and Zero
  - $1/H(s_p) = 0$, for the complex number s_p at a location of pole in s-plane
  - $H(s_z) = 0$, for the complex number s_z at a location of zero in s-plane
- Corner frequency of pole and zero
  - A corner frequency in Bode diagram is observed as a consequence of pole and zero.
  - Pole frequency: The corner of amplitude response is convex downward.
  - Zero frequency: The corner of amplitude response is convex upward.

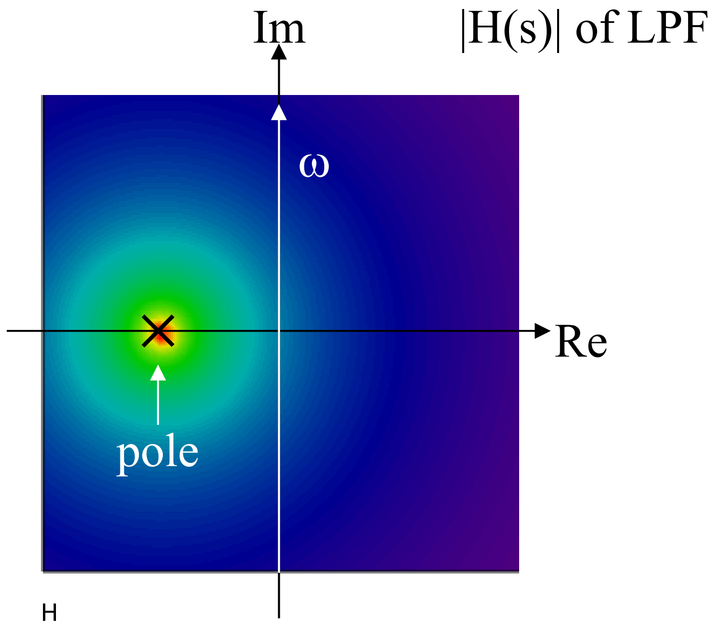
1-pole transfer function

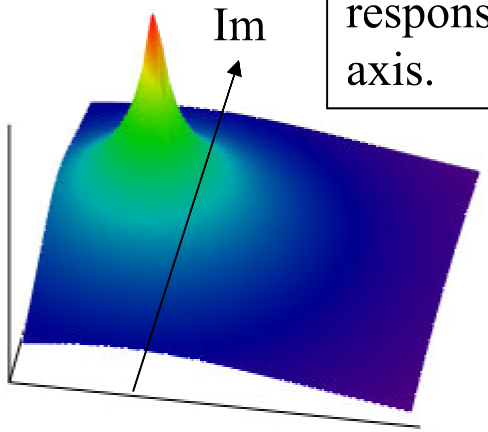
$$H(s) = \frac{a \cdot s + b}{s + c} = \frac{b}{c} \frac{(1 + \frac{as}{b})}{(1 + \frac{s}{c})}$$

(a, b, c = real number)

Type of frequency response

$a = 0, \quad b \neq 0$	LPF
$a \neq 0, \quad b = 0$	HPF





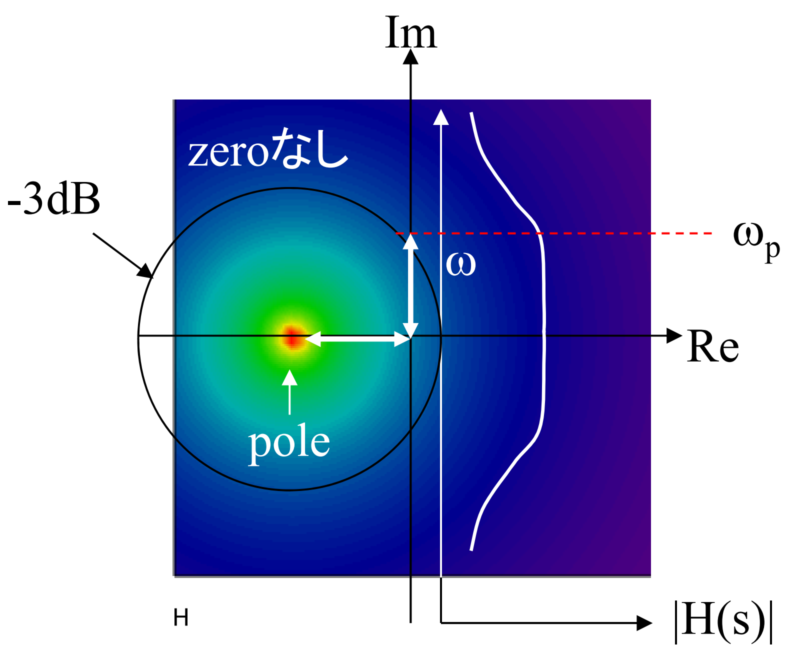
The Bode diagram is represents the amplitude response in the imaginary axis.

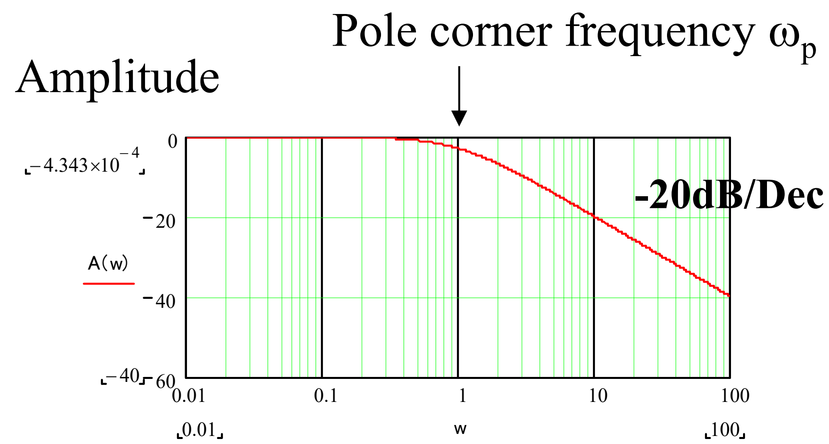
H

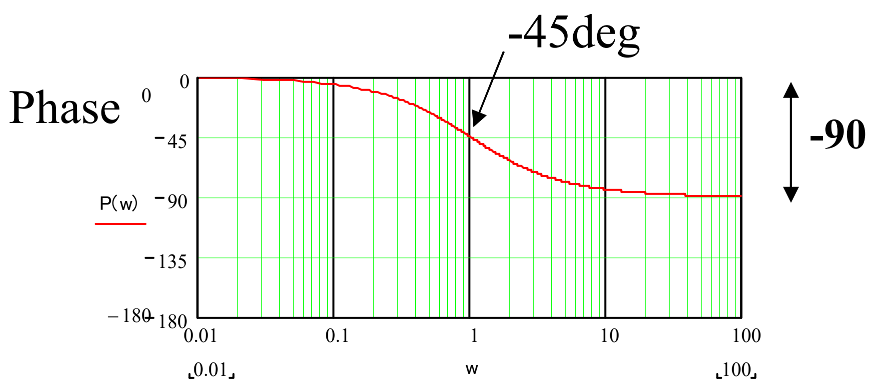
Bode diagram of 1st order LPF

Transfer function

$$H(s) = \frac{b}{s+c}$$





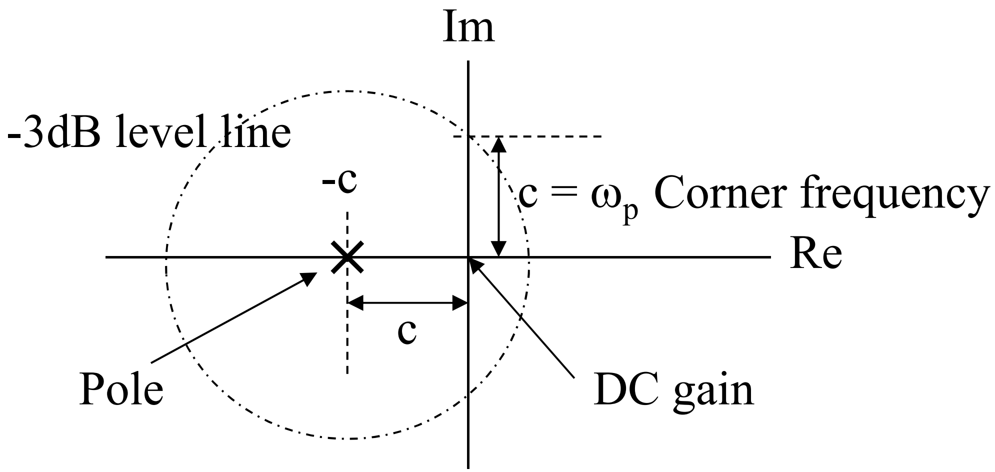


Positional relation between pole and corner frequency

$$H(s) = \frac{b}{s+c}$$

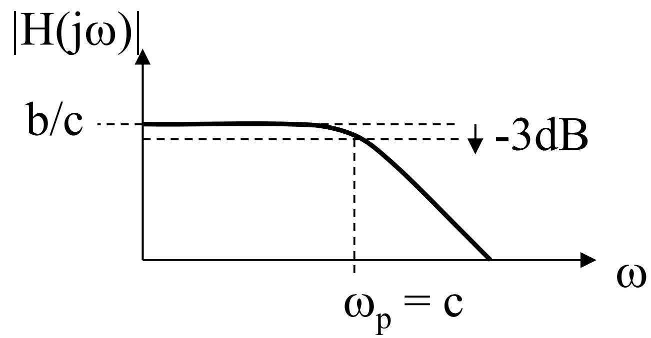
$$H(j\omega) = \frac{\frac{b}{c}}{1 + j\frac{\omega}{c}}$$

$$|H(j\omega)| = \frac{\frac{b}{c}}{\sqrt{1 + \frac{\omega^2}{c^2}}}$$



$$\begin{cases} \omega << c \rightarrow |H(j\omega)| = \frac{b}{c} \\ \omega = c \rightarrow |H(j\omega)| = \frac{1}{\sqrt{2}}\frac{b}{c} \\ \omega >> c \rightarrow |H(j\omega)| = \frac{b}{\omega} \end{cases}$$

$$\frac{1}{\sqrt{2}} = -3\text{dB}$$



1-pole, 1-zero transfer function

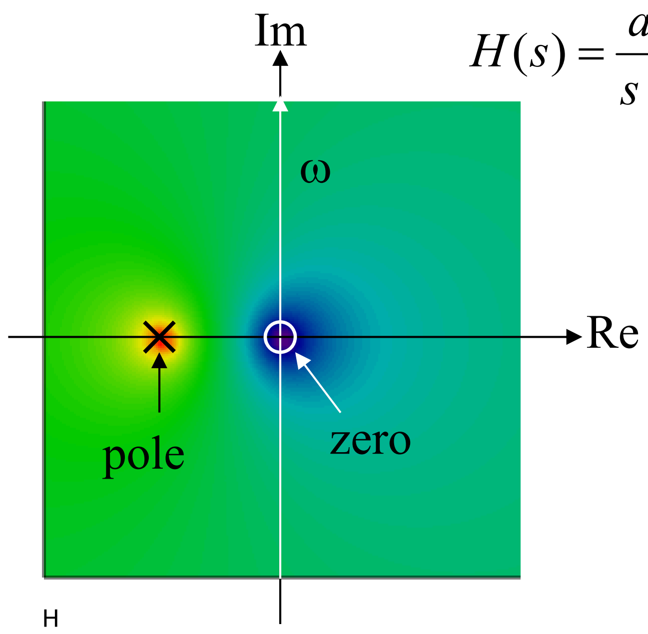
$$H(s) = \frac{a \cdot s + b}{s + c}$$ (a, b, c = real number)

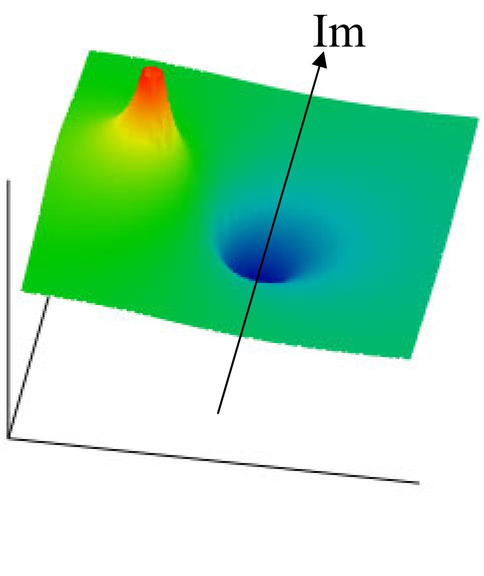
Type of frequency response

$a = 0, \quad b \neq 0$	LPF
$a \neq 0, \quad b = 0$	HPF

|H(s)| of HPF

$$H(s) = \frac{a \cdot s}{s + c}$$

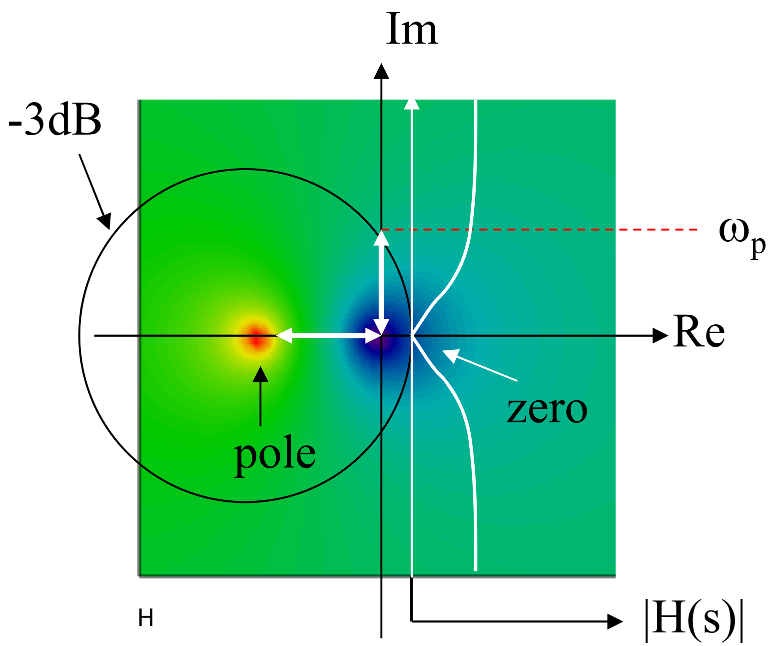


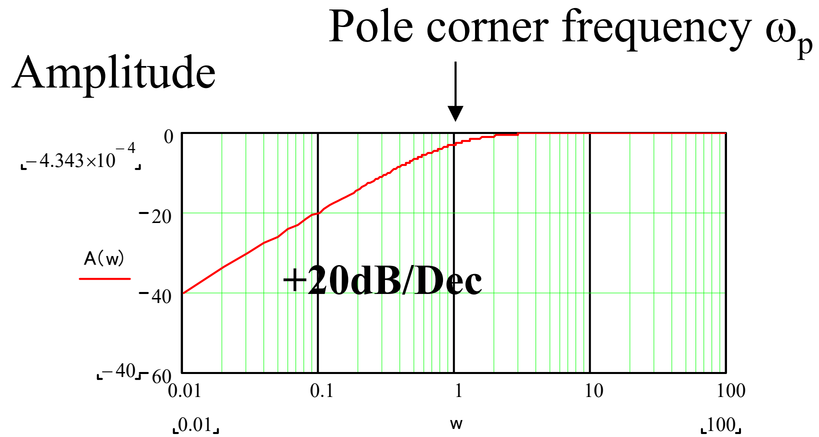


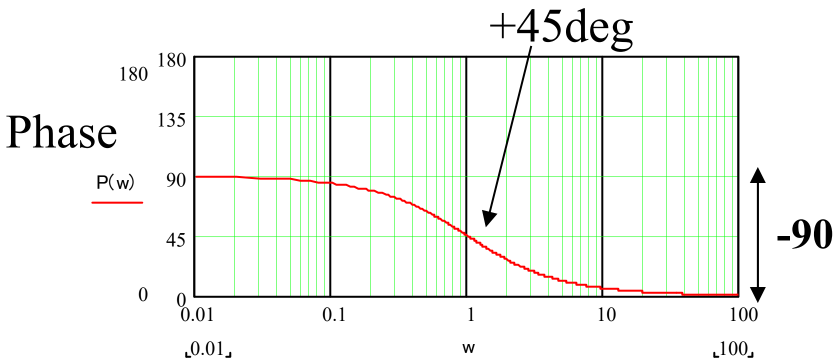
Bode diagram of 1st order HPF

Transfer function

$$H(s) = \frac{a \cdot s}{s + c}$$





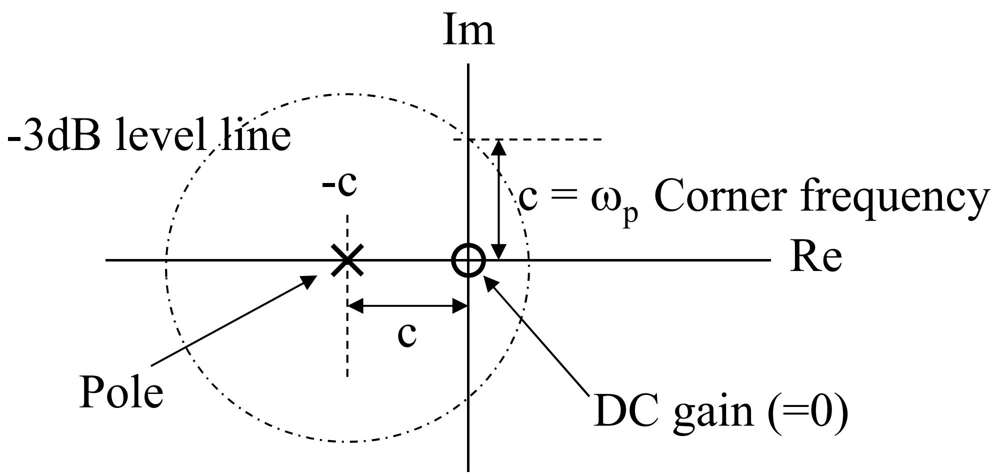


Positional relation between pole and corner frequency

$$H(s) = \frac{as}{s+c}$$

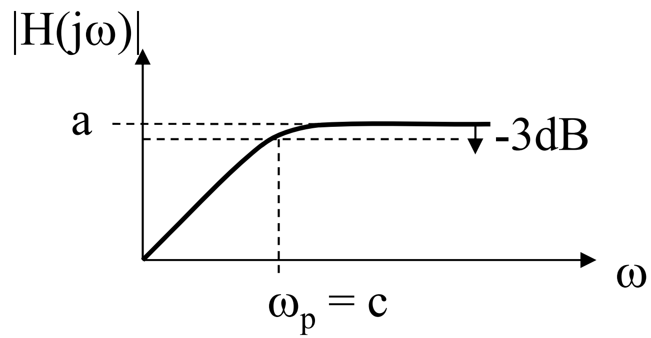
$$H(j\omega) = \frac{j\omega \frac{a}{c}}{1 + j\frac{\omega}{c}}$$

$$|H(j\omega)| = \frac{\omega \frac{a}{c}}{\sqrt{1 + \frac{\omega^2}{c^2}}}$$



$$\begin{cases} \omega << c \rightarrow |H(j\omega)| = \omega \frac{a}{c} \\ \omega = c \rightarrow |H(j\omega)| = \frac{a}{\sqrt{2}} \\ \omega >> c \rightarrow |H(j\omega)| = a \end{cases}$$

$$\frac{1}{\sqrt{2}} = -3\text{dB}$$



2-pole transfer function

$$H(s) = \frac{a \cdot s^2 + b \cdot s + c}{s^2 + d \cdot s + e}$$

(a, b, c = real number)

|H(s)| of LPF

$$H(s) = \frac{c}{s^2 + d \cdot s + e}$$

Solution of the pole

$$D(s) = s^2 + d \cdot s + e = 0$$

$$s = -\frac{d}{2} \pm j\sqrt{e - \frac{d^2}{4}}$$

Complex number of 2 poles

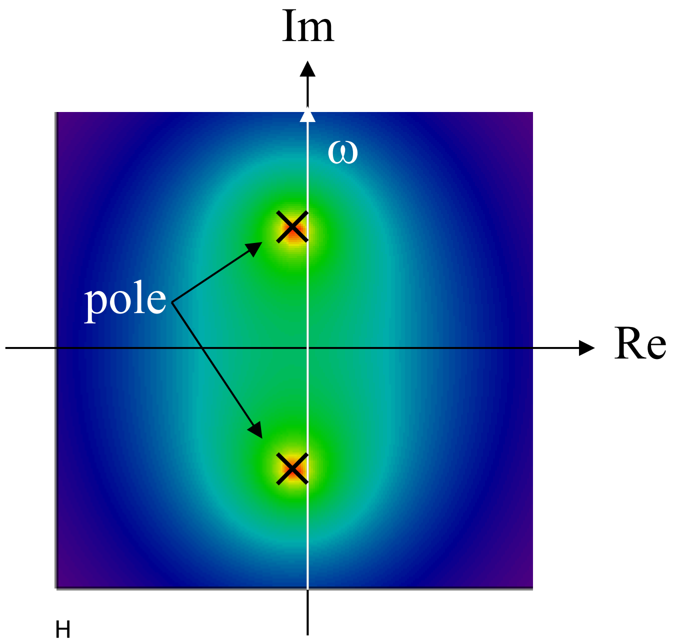
Type of frequency response

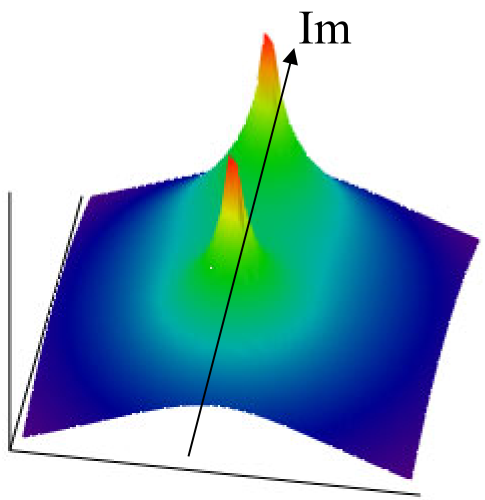
$a = b = 0, \quad c \neq 0$	LPF
$a = c = 0, \quad b \neq 0$	BPF
$b = c = 0, \quad a \neq 0$	HPF
$b = 0, \quad a \neq 0, \quad c \neq 0$	BEF

Note: If the denominator is factorable, the transfer function has 2 real poles.

Bode diagram of 2nd order LPF

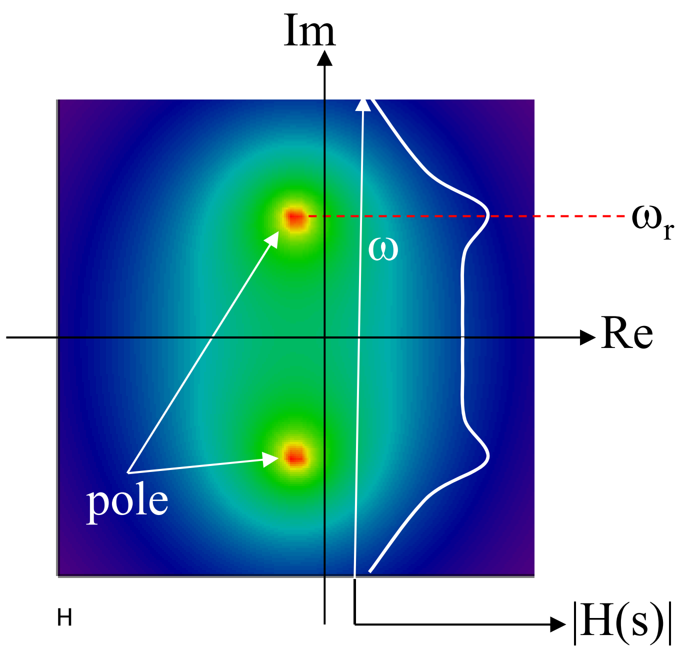
$$H(s) = \frac{c}{s^2 + d \cdot s + e}$$ の場合

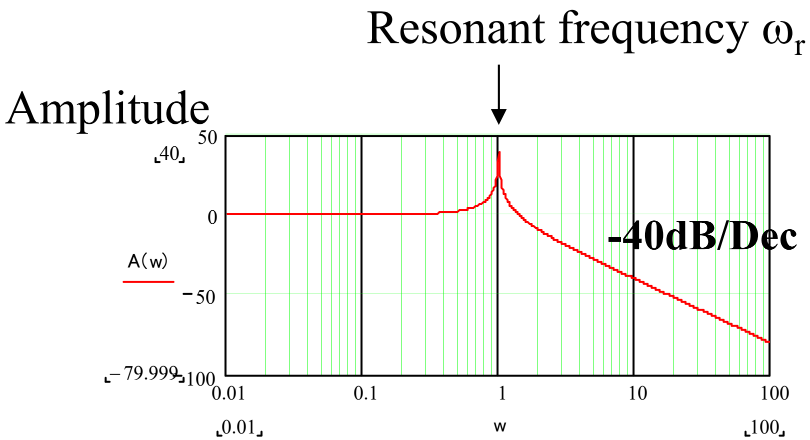


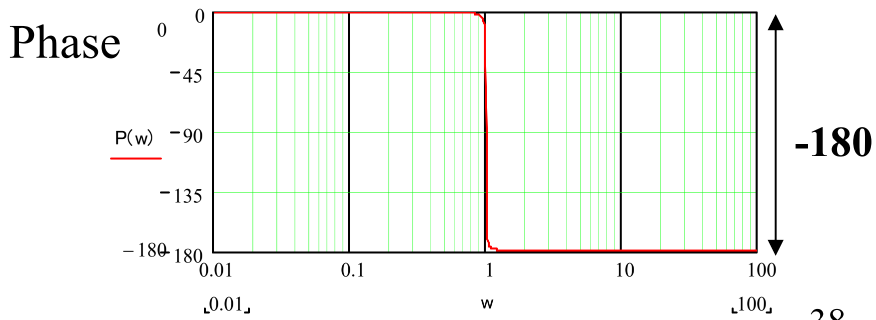


Bode diagram of 2nd order LPF

$$H(s) = \frac{c}{s^2 + d \cdot s + e}$$







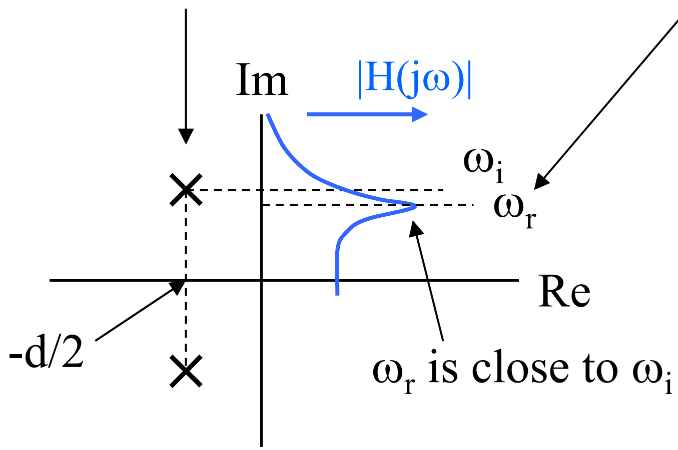
Positional relation between pole and corner frequency

$$H(s) = \frac{c}{s^2 + d \cdot s + e}$$

$s^2 + d \cdot s + e = 0$　のとき

$$s = \underbrace{-\frac{d}{2} \pm j\sqrt{e - \frac{d^2}{4}}}_{} \equiv -\frac{d}{2} \pm j\omega_i$$

Location of the pole in s-plane



$$|H(j\omega)| = \frac{c}{\sqrt{\{(j\omega)^2 + d(j\omega) + e\}\{(-j\omega)^2 + d(-j\omega) + e\}}}$$

$$= \frac{c}{\sqrt{\{\omega^2 - (e - \frac{d^2}{2})\}^2 + d^2(e - \frac{d^2}{4})}}$$

$$\omega_r \equiv \sqrt{(e - \frac{d^2}{2})}$$

$$|H(j\omega)| = \frac{c}{\sqrt{\{\omega^2 - \omega_r^2\}^2 + d^2(e - \frac{d^2}{4})}}$$

If $\omega = \omega_r$, the amplitude reach a maximum.

$$|H(j\omega_r)| = \frac{c}{d\sqrt{e - \frac{d^2}{4}}}$$

The smaller d causes the higher peak.

1.5 Transfer function of discrete-time analog/digital circuits

Definition of transfer function on z-plane

$$H(s) = \frac{Output\ signal\ (s)}{Input\ signal\ (s)}$$

$$z^{-1} = e^{-sT_s} \quad \downarrow$$

$$H(z) = \frac{Output\ signal\ (z)}{Input\ signal\ (z)}$$

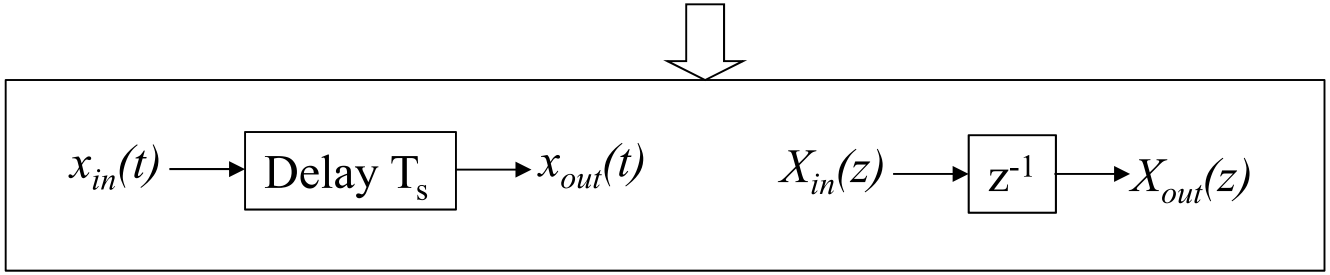
The non-linear functions of z-variable are deduced from the rational functions including an integration and a differentiation of s-variable. The complexity of the circuit implementation is avoided by using a translation theorem and some approximations.

Translation Theorem in z-transform

$$x(nT_s) \xrightarrow{Z} X(z) \qquad (1)$$

$$x(nT_s + mT_s) \xrightarrow{Z} z^{m} X(z) - z^{m} \sum_{n=0}^{m-1} x(nT_s) z^{-n} \qquad (2)$$

$$x(nT_s - mT_s) \xrightarrow{Z} z^{-m} X(z) \qquad (3)$$



The equation 3 shows that the multiplication of z^{-m} in z-plane is equivalent to the delay of mT_s in time domain. The z^{-1} operator is called "Delay element".

Approximation of Z transform 1

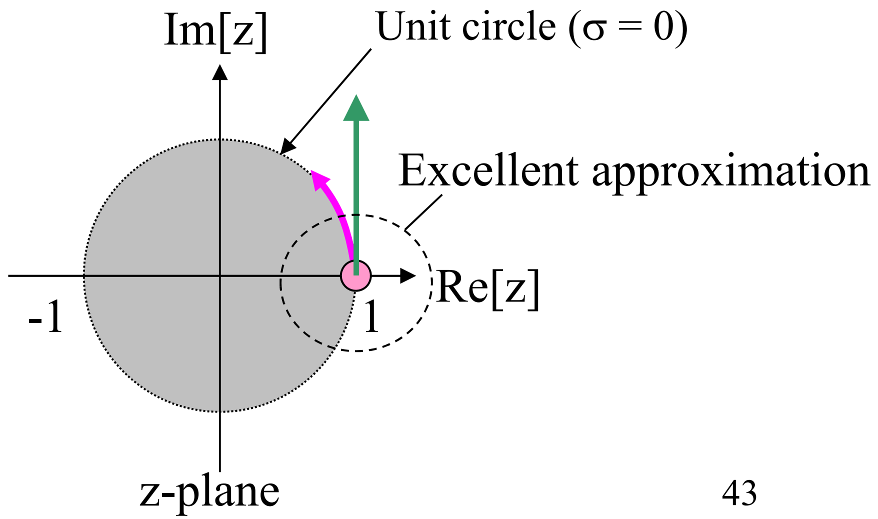
- Forward Euler Transformation (FET)

Power series expansion of z at s = 0

$$z = e^{sT_S} = 1 + \frac{1}{1!}(sT_S)^1 + \frac{1}{2!}(sT_S)^2 + \frac{1}{3!}(sT_S)^3 + \Lambda$$

$$\approx 1 + sT_S$$

$$\therefore s \approx \frac{z-1}{T_S}$$



Approximation of Z transform 2

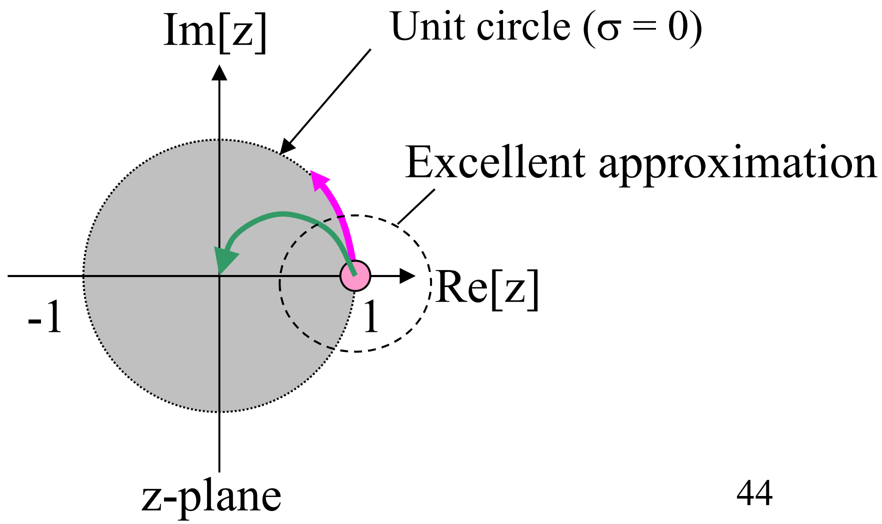
- Backward Euler Transformation (BET)

Power series expansion of z^{-1} at s = 0

$$z^{-1} = e^{-sT_S} = 1 - \frac{1}{1!}(sT_S)^1 + \frac{1}{2!}(sT_S)^2 - \frac{1}{3!}(sT_S)^3 + \Lambda$$

$$\approx 1 - sT_S$$

$$\therefore s \approx \frac{1 - z^{-1}}{T_S}$$



Approximation of Z transform 3

- Bilinear Transformation

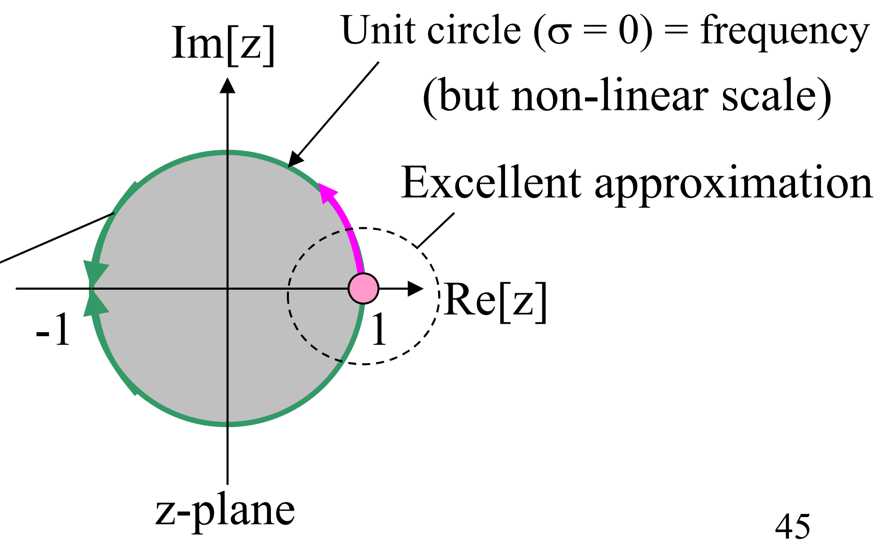
Power series expansion of $\log_e z$ at s = 0

$$sT_S = \ln z = 2 \cdot [\frac{z-1}{z+1} + \frac{1}{3}\frac{(z-1)^3}{(z+1)^3} + \frac{1}{5}\frac{(z-1)^5}{(z+1)^5} + \Lambda]$$

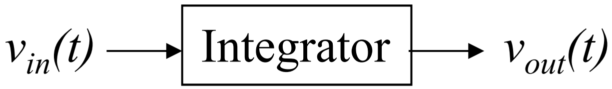
$$\approx 2\frac{z-1}{z+1}$$

$$\therefore s \approx \frac{2}{T_S}\frac{z-1}{z+1} = \frac{2}{T_S}\frac{1-z^{-1}}{1+z^{-1}}$$

$$\omega_{bilinear} = \frac{2}{T_s} arctan \frac{\omega T_s}{2}$$



Integration of continuous-time signal



$$v_{out}(t) = \int_0^t v_{in}(\tau)d\tau$$

⇩ Laplace transform

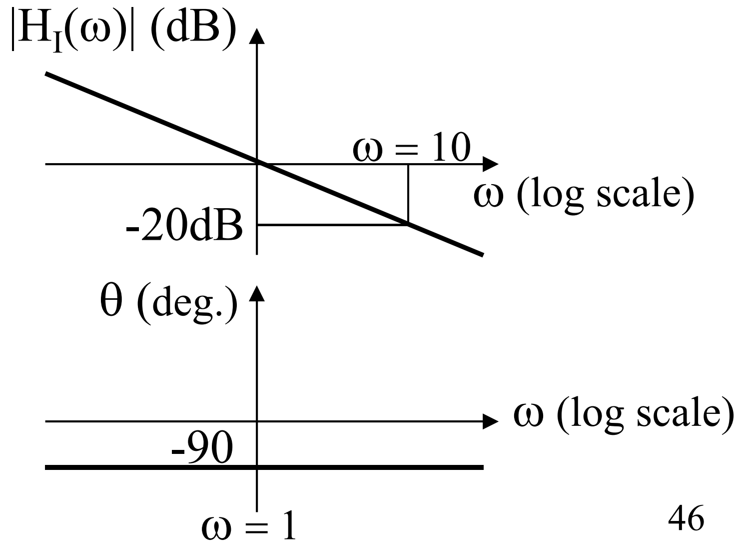
$$V_{out}(s) = \frac{1}{s}V_{in}(s) + \frac{1}{s}\int_{-\infty}^{0} v_{in}(\tau)d\tau$$

$$= \frac{1}{s}V_{in}(s) \quad \text{(Periodic function)}$$

Frequency transfer function (s = jω)

$$H_I(\omega) = \frac{1}{s} = \frac{1}{j\omega} = -j\frac{1}{\omega}$$

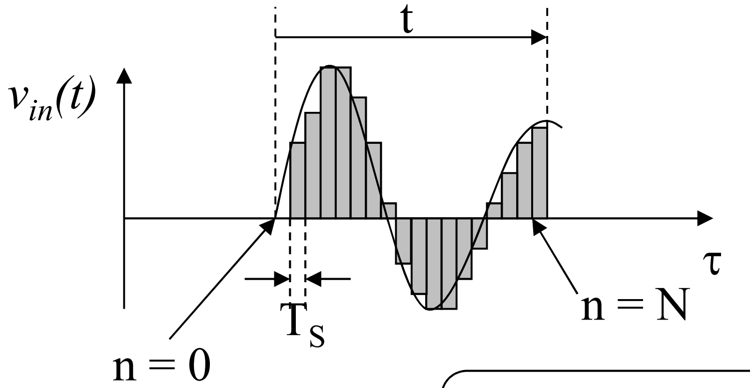
$$|H_I(\omega)|[dB] = 20\log|H_I(\omega)| = 20\log\frac{1}{\omega} = -20\log\omega$$



Integration of discrete-time signal

Integration approximated with BET

$$H_I(s) = \frac{1}{s} \cong \frac{T_S}{1 - z^{-1}}$$



$$v_{out}(t) = \int_0^t v_{in}(\tau) d\tau$$

Discretized

$$v_{out}(t) = \sum_{n=0}^{N} v_{in}(t - nT_S) \cdot T_S \quad \overset{Z}{\Longrightarrow} \quad V_{out}(z) = T_S \sum_{n=0}^{N} z^{-n} V_{in}(z)$$

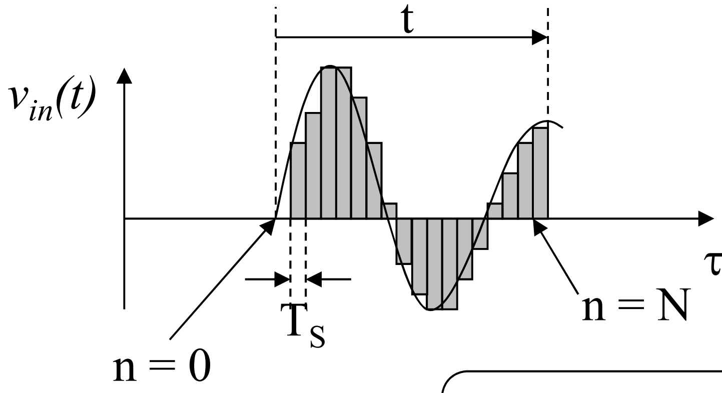
Geometrical series of z^{-1}

$$\xrightarrow{N=\infty} \underbrace{\frac{T_S}{1 - z^{-1}}}_{H_I(s)} V_{in}(z)$$

Integration of discrete-time signal

Integration approximated with BET

$$H_I(s) = \frac{1}{s} \cong \frac{T_S}{1 - z^{-1}}$$



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Discretized

$$v_{out}(t) = \sum_{n=0}^{N} v_{in}(t - nT_S) \cdot T_S \quad \overset{Z}{\Longrightarrow} \quad V_{out}(z) = T_S \sum_{n=0}^{N} z^{-n} V_{in}(z)$$

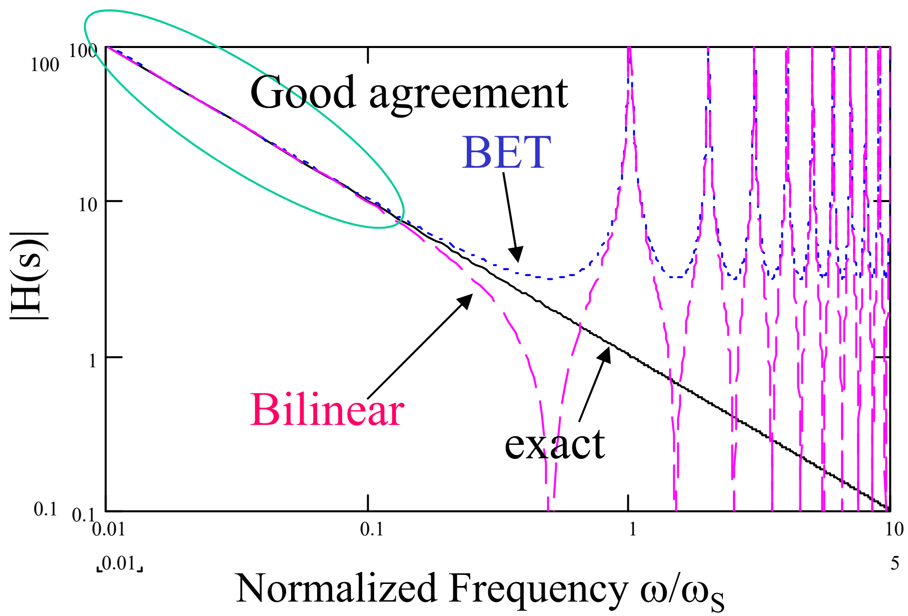
Geometrical series of z^{-1}

$$\xrightarrow{N=\infty} \underbrace{\frac{T_S}{1 - z^{-1}}}_{H_I(s)} V_{in}(z)$$

Integration error due to approximation

Frequency transfer function of integrator $H(s) = \frac{1}{s} = -j\frac{1}{\omega}$

$$s = 0 \rightarrow \quad z^{-1} = e^{-sT_S} = e^{-j2\pi\frac{\omega}{\omega_S}}$$



BET of Integrator

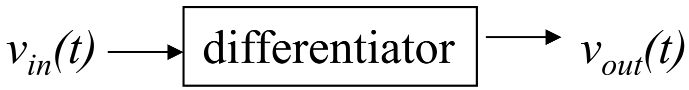
$$\frac{T_S}{1-z^{-1}} = \frac{T_S}{1-e^{-j\omega T_S}} = \frac{T_S}{1-e^{-j2\pi\frac{\omega}{\omega_S}}}$$

Bilinear transformation of integrator

$$\frac{T_S}{2}\frac{1+z^{-1}}{1-z^{-1}} = \frac{T_S}{2}\frac{1+e^{-j\omega T_S}}{1-e^{-j\omega T_S}} = \frac{T_S}{2}\frac{1+e^{-j2\pi\frac{\omega}{\omega_S}}}{1-e^{-j2\pi\frac{\omega}{\omega_S}}}$$

The approximation is excellent in $\omega \ll \omega_s/2$.

Differentiation of continuous-time signal



$$v_{out}(t) = \frac{d}{dt} v_{in}(\tau)$$

⇓ Laplace transform

$$V_{out}(s) = sV_{in}(s) - v_{in}(0)$$

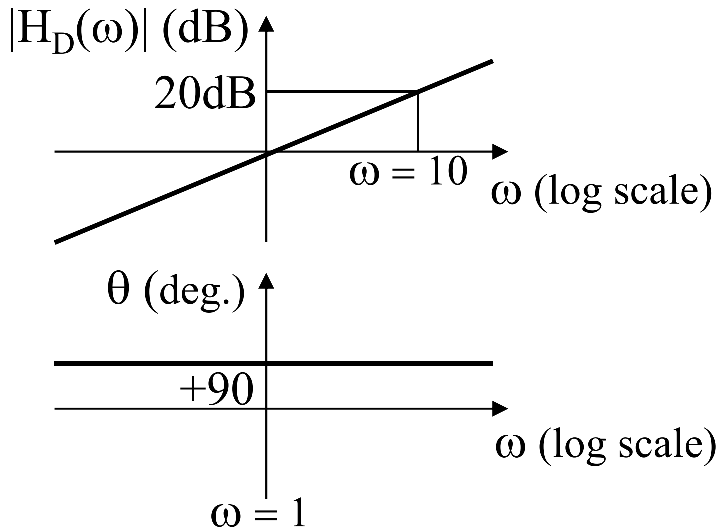
$$V_{out}(s) = sV_{in}(s)$$

（t = 0 で信号がない場合）

Frequency transfer function (s = jω)

$$H_D(s) = s = j\omega$$

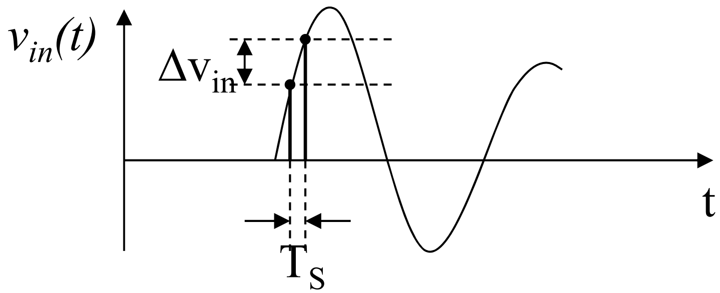
$$|H_D(\omega)|[dB] = 20\log|H_D(\omega)| = 20\log\omega$$



Differentiation of discrete-time signal

Differentiation approximated with BET

$$H_D(s) = s \cong \frac{1 - z^{-1}}{T_S}$$



$$v_{out}(t) = \frac{d}{dt} v_{in}(\tau)$$

⇓ Discretized

$$v_{out}(t) = \frac{v_{in}(t) - v_{in}(t - T_S)}{T_S}$$

Z ⇒

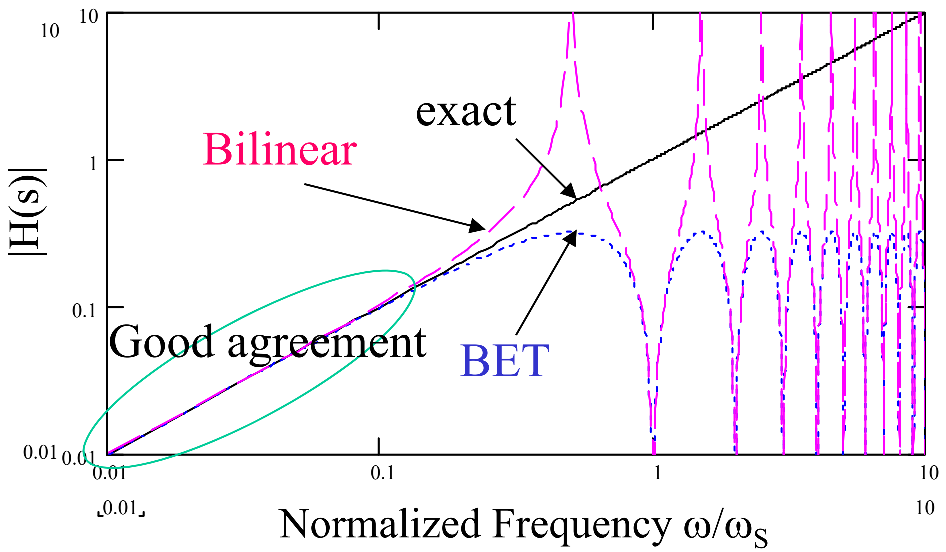
$$V_{out}(z) = \frac{V_{in}(z) - z^{-1} V_{in}(z)}{T_S} = \underbrace{\frac{1 - z^{-1}}{T_S}}_{H_D(s)} V_{in}(z)$$

Delay element

Differentiation error due to approximation

Frequency transfer function of integrator $\quad H(s) = s = j\omega$

$s = 0 \rightarrow \quad z^{-1} = e^{-sT_S} = e^{-j2\pi\frac{\omega}{\omega_S}}$



BET of differentiator

$$\frac{1-z^{-1}}{T_S} = \frac{1-e^{-j\omega T_S}}{T_S} = \frac{1-e^{-j2\pi\frac{\omega}{\omega_S}}}{T_S}$$

Bilinear transformation of differentiator

$$\frac{2}{T_S}\frac{1-z^{-1}}{1+z^{-1}} = \frac{2}{T_S}\frac{1-e^{-j\omega T_S}}{1+e^{-j\omega T_S}} = \frac{2}{T_S}\frac{1-e^{-j2\pi\frac{\omega}{\omega_S}}}{1+e^{-j2\pi\frac{\omega}{\omega_S}}}$$

The approximation is excellent in $\omega \ll \omega_s/2$.