2. Transfer function

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2.1 Definition of transfer function

Definition of transfer function

$$V_{IN} \longrightarrow \boxed{F(s) \text{ or } F(z)} \longrightarrow V_{OUT}$$

- Transfer function of CT circuit

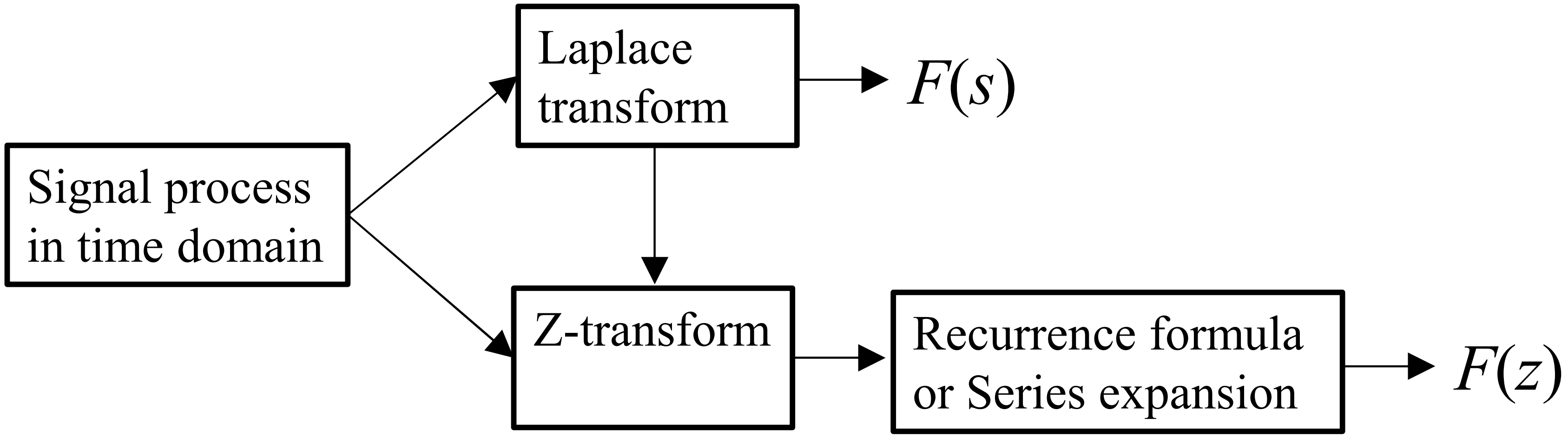
$$F(s) = \frac{V_{OUT}(s)}{V_{IN}(s)}$$

- Transfer function of DT circuit

$$F(z) = \frac{V_{OUT}(z)}{V_{IN}(z)}$$

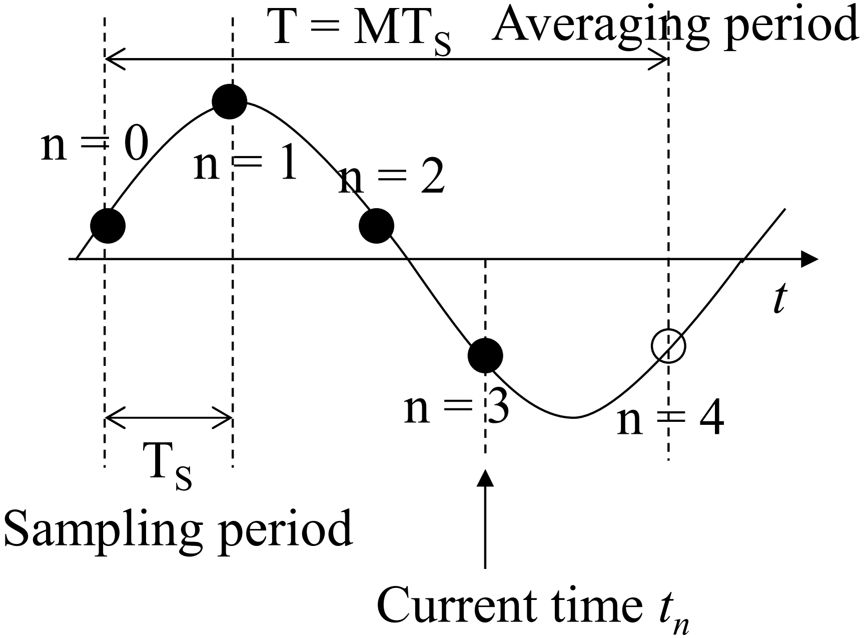
[NOTE] A transfer function is defined in s-plane or z-plane.

Method for deriving transfer function



Transfer function in z-domain

Transfer function of moving average
In the case of M = 4



Signal process in time domain

$$y(t_n) = \frac{1}{T}\sum_{n=0}^{M-1} x(t_n)T_S = \frac{1}{M}\sum_{n=0}^{M-1} x(t_n)$$

$$= \frac{1}{M}\{x(t_0) + x(t_1) + x(t_2) + x(t_3)\}$$

$$\downarrow \mathcal{Z}$$

Series expansion of z variable

$$Y(z) = \frac{1}{M}\sum_{n=0}^{M-1} z^{-n}X(z)$$

$$= \frac{1}{M}\left(1 + z^{-1} + \cdots + z^{-(M-1)}\right)X(z)$$

$$MY(z) = (z^{-1}+z^{-2} + \cdots +z^{-M})X(z) + (z^0 - z^{-M})X(z) = z^{-1}MY(z) + (1 - z^{-M})X(z)$$

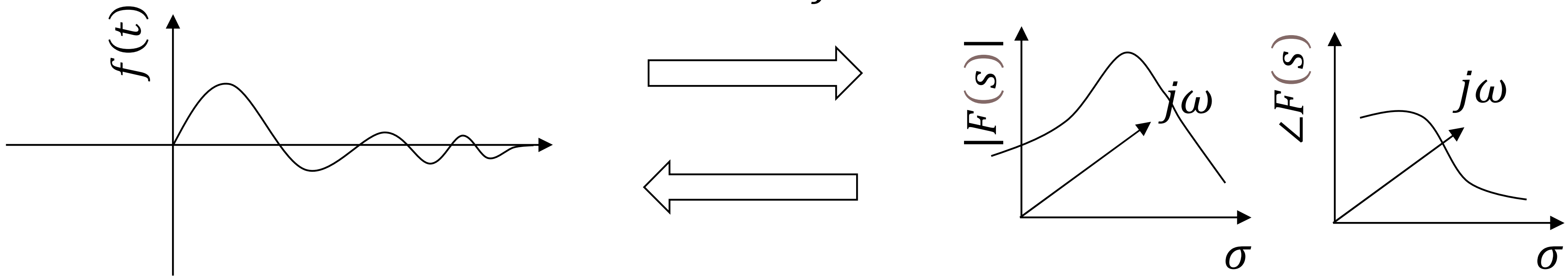
$$(1 - z^{-1})MY(z) = (1 - z^{-M})X(z) \longrightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1}{M}\frac{(1 - z^{-M})}{(1 - z^{-1})}$$

Definition of Laplace transform

Laplace transform

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t) e^{-st} dt$$

$$s = \sigma + j\omega$$

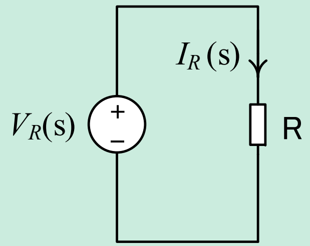
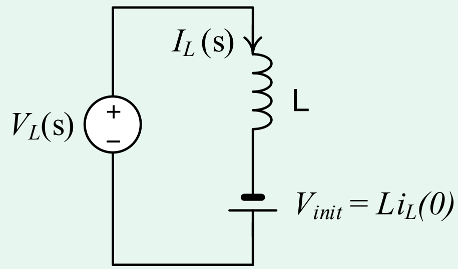
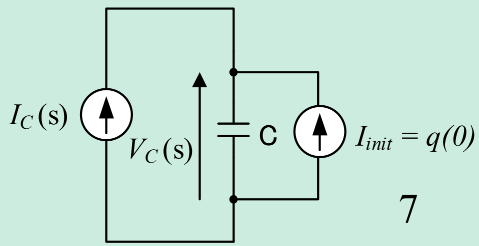


Inverse Laplace transform

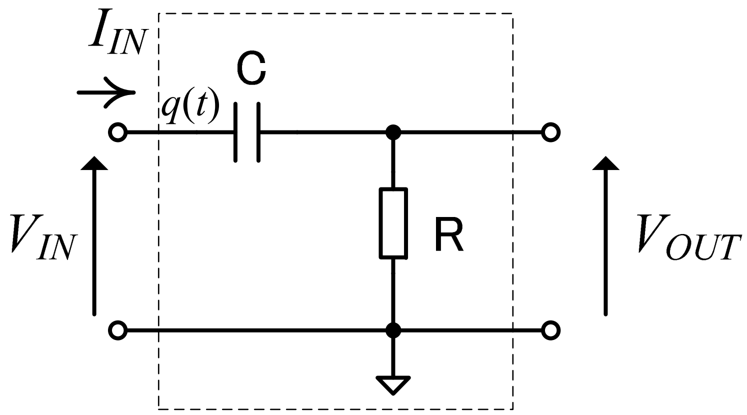
$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} ds$$

Note: Current–voltage characteristic

Circuit equations are derived from Kirchhoff's law and current-voltage (I-V) characteristic for each device. Therefore, it will be useful if you know about Laplace transformed current-voltage characteristics in advance.

Device	I-V characteristic in t-domain	I-V characteristics in s-domain	Equivalent circuits
R	$v_R(t) = R \cdot i_R(t)$	$V_R(s) = R \cdot I_R(s)$	
L	$v_L(t) = L\dfrac{di_L(t)}{dt}$	$V_L(s) + L \cdot i_L(0) = sL \cdot I_L(s)$	
C	$v_C(t) = \dfrac{1}{C}\displaystyle\int_0^t i_C(\tau)d\tau$	$V_C(s) = \dfrac{1}{sC}\{I_C(s) + q(0)\}$	

Transfer function in s-domain



$$
\begin{cases}
v_{in}(t) = \dfrac{1}{C}\displaystyle\int_0^{\infty} i_{in}(t)dt + Ri_{in}(t) \\[2ex]
v_{out}(t) = Ri_{in}(t)
\end{cases}
$$

$$\Big\downarrow \; \mathcal{L} \quad \text{(Initial condition: } q(0) \text{)}$$

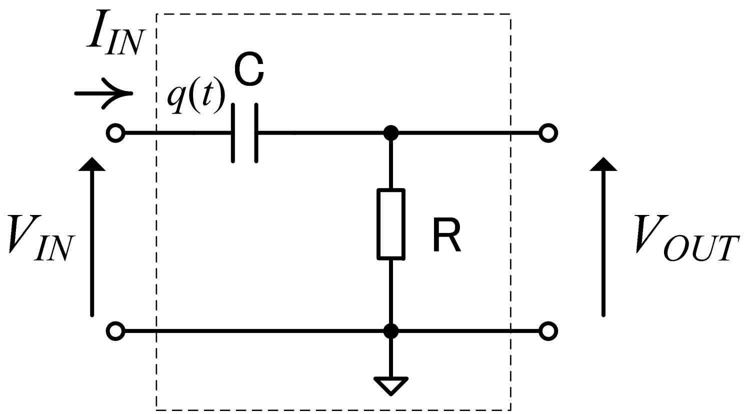
$$
\begin{cases}
V_{IN}(s) = \dfrac{1}{C}\left(\dfrac{I_{IN}(s)}{s} + \dfrac{q(0)}{s}\right) + RI_{IN}(s) \\[2ex]
V_{OUT}(s) = RI_{IN}(s)
\end{cases}
$$

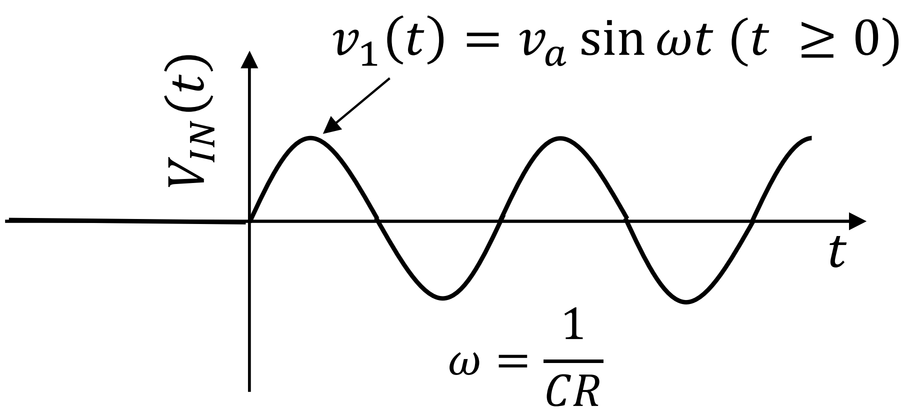
$$V_{OUT}(s) = \frac{CR}{1 + sCR} V_{IN}(s) - \frac{R}{1 + sCR} q(0)$$

Transfer function for input signal

Transfer function for initial condition

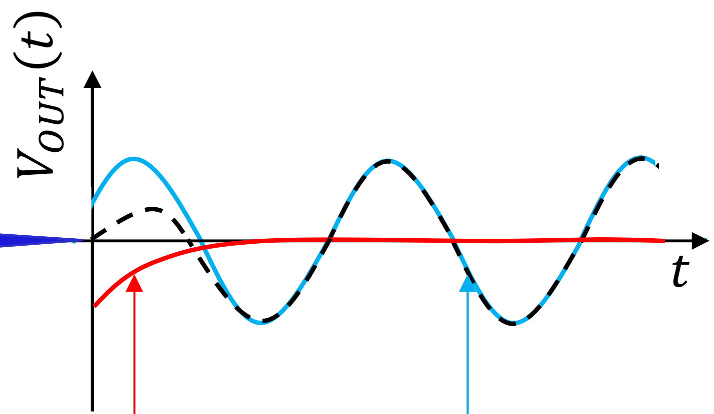
Transient and steady state response





Initial condition: $q(0) = 0$

The transient response is generated by changing the waveform of input signal.



$$V_{OUT}(s) = \frac{R}{\frac{1}{sC} + R} V_{IN}(s) \xrightarrow{\mathcal{L}^{-1}} v_{out}(t) = \underbrace{\boxed{-\frac{v_a}{2} e^{-\omega t}}}_{\text{Transient}} + \underbrace{\boxed{\frac{v_a}{2}(\cos \omega t + \sin \omega t)}}_{\text{Steady state}}$$

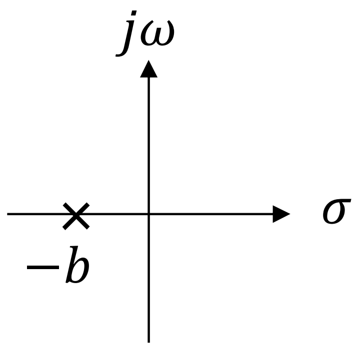
（参考）$j\omega$とラプラス変換

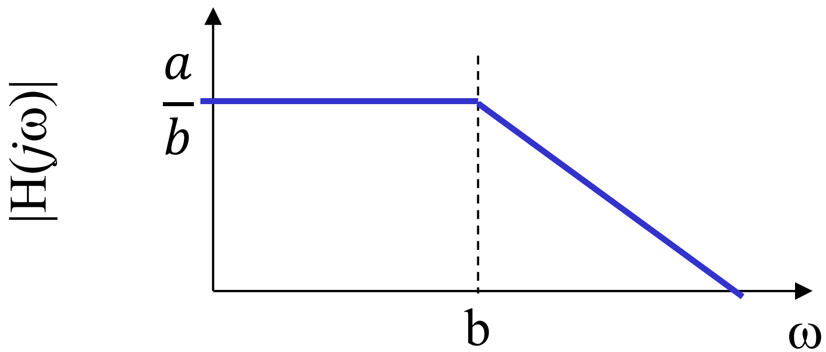
- 複素ベクトルによる解析
  - 周期的入力波形に対する定常解のみが得られる。
  - ラプラス変換された回路方程式で、初期条件$= 0, s = j\omega$とすると複素ベクトルによる解析と同じ結果が得られる。
- ラプラス変換による解析
  - 入力信号に対する応答と初期値に対する応答の重ね合わせが得られる。（スライド7の例）
  - 初期値 $= 0$ とすると、入力信号に対する過渡応答が得られる。
  - 定常解を得るためには、$t = 0$における値（$t < 0$における定常解が必要）を求めて初期条件とするか、$t = \infty$の解を求める必要がある。（スライド8の例）

Transfer function in frequency domain

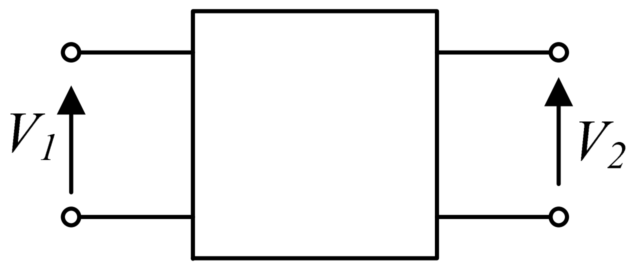
- The frequency domain transfer function (or the frequency response) is obtained from the transfer function under the condition that $s = j\omega$.
- The frequency transfer function means the steady state characteristic of the circuit which is stimulated with $e^{j\omega t}$.

$$H(s) = \frac{a}{b+s} \xrightarrow{s=j\omega} H(j\omega) = \frac{a}{b+j\omega} = \frac{\frac{a}{b}}{1+j\frac{\omega}{b}}$$





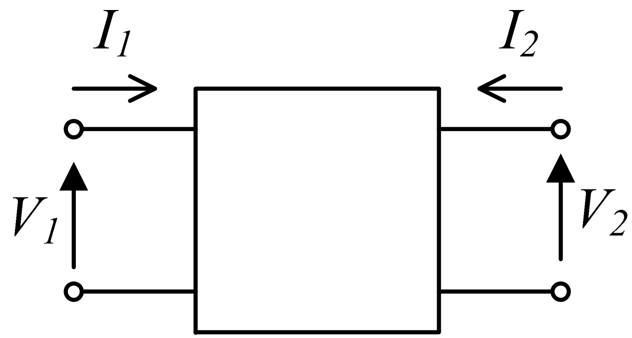
Two-port network parameters and transfer function



Transfer function

$$V_2(s) = H(s) \cdot V_1(s)$$

The transfer function varies in according with the load impedance.



Two-port network parameters

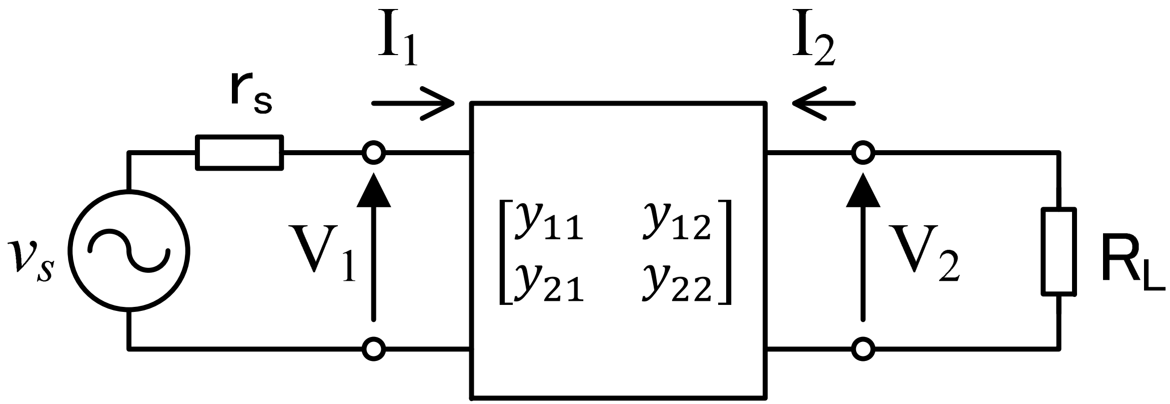
$$\begin{bmatrix} I_1(j\omega) \\ I_2(j\omega) \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1(j\omega) \\ V_2(j\omega) \end{bmatrix}$$

$$\begin{bmatrix} V_1(j\omega) \\ V_2(j\omega) \end{bmatrix} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1(j\omega) \\ I_2(j\omega) \end{bmatrix}$$

$$\begin{bmatrix} V_1(j\omega) \\ I_1(j\omega) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2(j\omega) \\ -I_2(j\omega) \end{bmatrix}$$

Two-port network parameters do not vary in according with the load impedance.

Load-dependence of transfer function

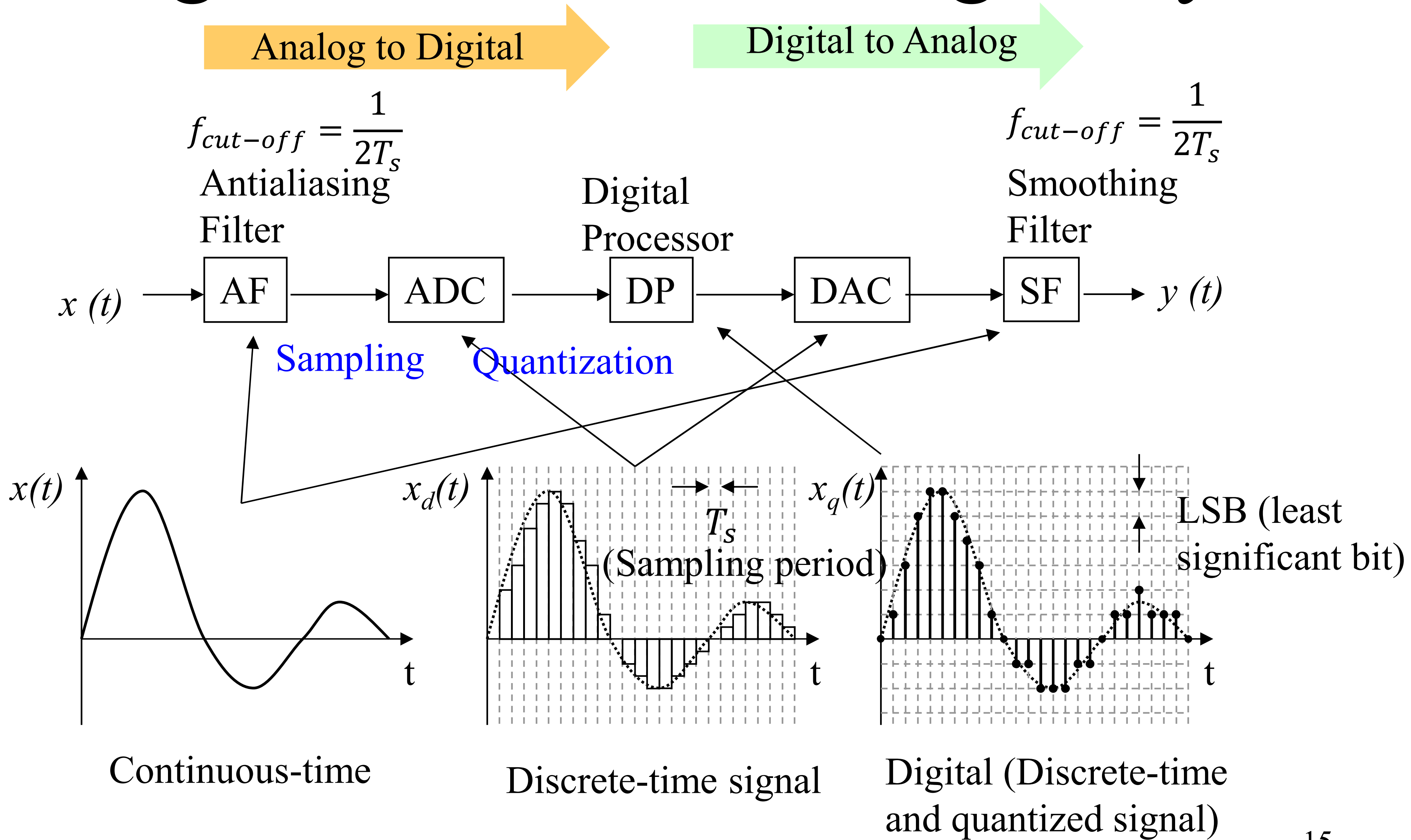


$$\begin{bmatrix} I_1(j\omega) \\ I_2(j\omega) \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1(j\omega) \\ V_2(j\omega) \end{bmatrix} \Rightarrow \begin{cases} I_2 = y_{21}V_1 + y_{22}V_2 \\ -V_2 = R_L I_2 \end{cases}$$

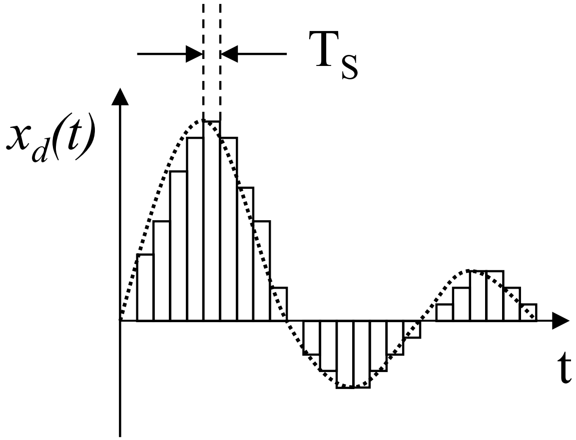
$$H(s = j\omega) = \frac{V_2}{V_1} = -\frac{y_{21}}{y_{22} + \frac{1}{R_L}}$$

2.2 Waveforms in mixed-signal circuits

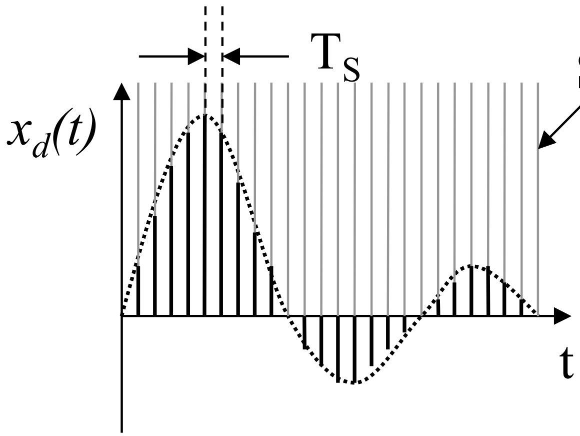
Configuration of mixed-signal system



Expressions of discrete-time signal



Step sampling
(Zero-order hold)



Impulse sampling
(Pulse Amplitude Modulation or PAM)

$$x_{du}(t) = \sum_n x(nT_S) \cdot \underbrace{\{u(t - nT_S) - u(t-(n+1)T_S)\}}_{\text{Step Sampler}}$$

$$x_d(t) = \sum_n x(t) \cdot \underbrace{\delta(t - nT_S)}_{\text{Impulse Sampler}}$$

NOTE: The discrete analog signal is practically obtained by S/H, but the signal can be handled similar to the impulse sequences (See slide 17).

Laplace transform and Z transform

Continuous-time signal *x(t)* $\xrightarrow{\mathcal{L}}$ $X(s) = \int_0^\infty x(t)e^{-st}dt$

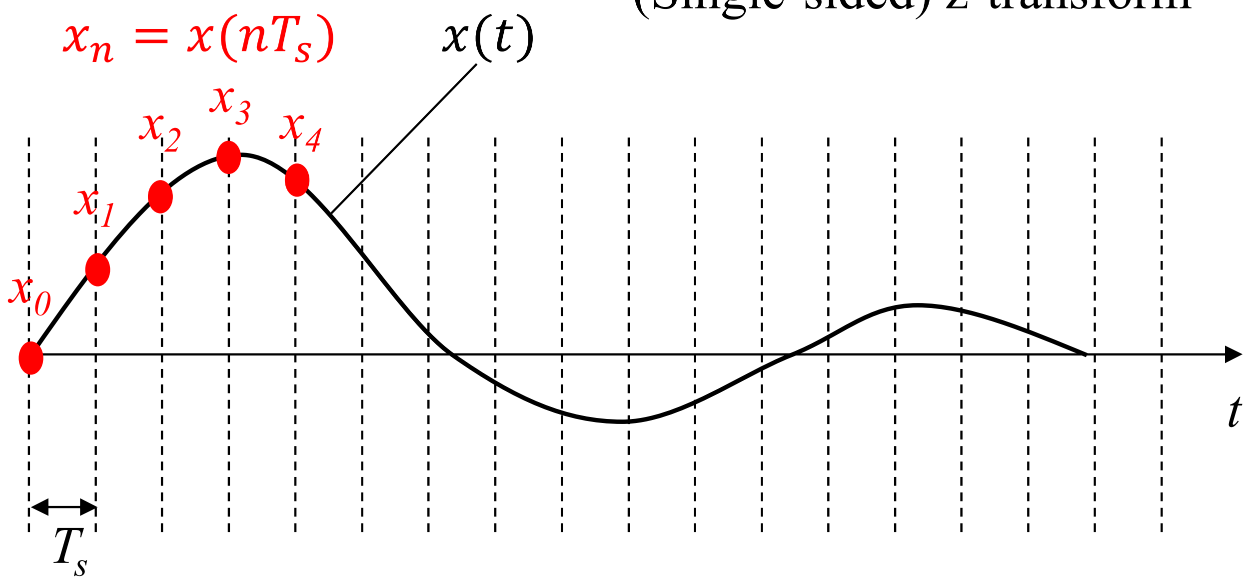
Discrete-time signal *x(t)* $\xrightarrow{\mathcal{Z}}$ $X(z) = \sum_{n=0}^{\infty} x_n z^{-n}$

(Single-sided) z-transform

$s = \sigma + j\omega$

$\Updownarrow$

$z^{-1} = e^{-sT_s}$



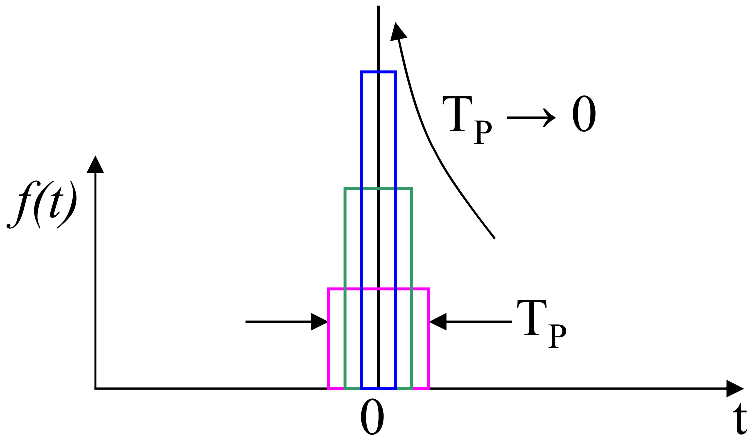
Characteristic of δ function

Step function $f(t)$

$$f(t) = \frac{1}{T_P}\{u(t+\frac{T_P}{2}) - u(t-\frac{T_P}{2})\}$$

$$\int_{-\infty}^{\infty} f(t)dt = 1 \qquad (1)$$

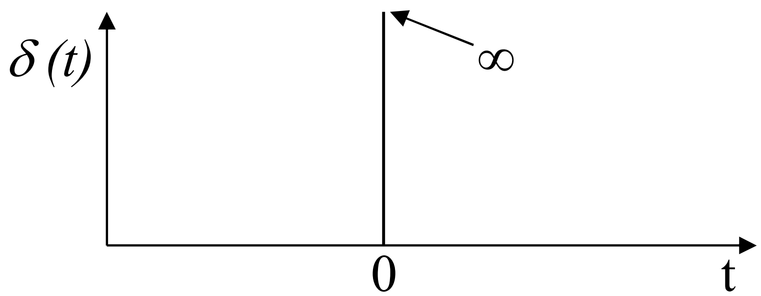
$$\delta(t) \equiv \lim_{T_P \to 0} f(t) \qquad (2)$$



Delta function δ

Important

$$\begin{cases} \int_{-\infty}^{\infty} \delta(t)dt = 1 & (3) \\ \delta(t) = 0 \quad (t \neq 0) & (4) \end{cases}$$



Z transform of discrete-time signal

PAM signal

$$x_d(t) = \sum_n x(t)\delta(t - nT_s)$$

$\Downarrow$ Laplace transform

$$X_d(s) = \int_0^\infty x_d(t)e^{-st}dt$$

$$= \int_0^\infty \sum_n x(t)\delta(t - nT_s)e^{-st}dt$$

$$= \sum_n \int_0^\infty x(t)\delta(t - nT_s)e^{-st}dt$$

$$= \sum_n x(nT_s)e^{-s(nT_s)}$$

$$z^{-1} = e^{-sT_s}$$

$$= \sum_n x(nT_s)z^{-n}$$

Zero-order hold signal

$$x_u(t) = \sum_n x(t)\{u(t - nT_s) - u(t - (n+1)T_s)\}$$

$\Downarrow$ Laplace transform

$$X_u(s) = \int_0^\infty x_u(t)e^{-st}dt$$

$$= \int_0^\infty \sum_n x(t)\{u(t - nT_s) - u(t - (n+1)T_s)\}e^{-st}dt$$

$$= \sum_n x(nT_s)\int_0^\infty \{u(t - nT_s) - u(t - (n+1)T_s)\}e^{-st}dt$$

$$= \sum_n x(nT_s)\frac{e^{-snT_s}(1 - e^{-sT_s})}{s}$$

$$z^{-1} = e^{-sT_s}$$

$$= \underbrace{\frac{1 - e^{-sT_s}}{s}}_{\text{Transfer function of step sampler}} \underbrace{\sum_n x(nT_s)z^{-n}}_{\text{PAM signal}}$$

See Appndix 2(3). ← Transfer function of step sampler

Z transform of PAM signal

Continuous-time signal $\quad x(t) \xrightarrow{\text{Laplace transform}} X(s) \xrightarrow{z = e^{sT_s}} X(z)$

$\downarrow$ Impulse sampling

Discrete-time signal $\quad x_d(t) = \sum_n x(nT_s)\delta(t - nT_s)$

$\downarrow$ Z transform

$$X_d(z) = \sum_n x_n z^{-n} \qquad (5)$$

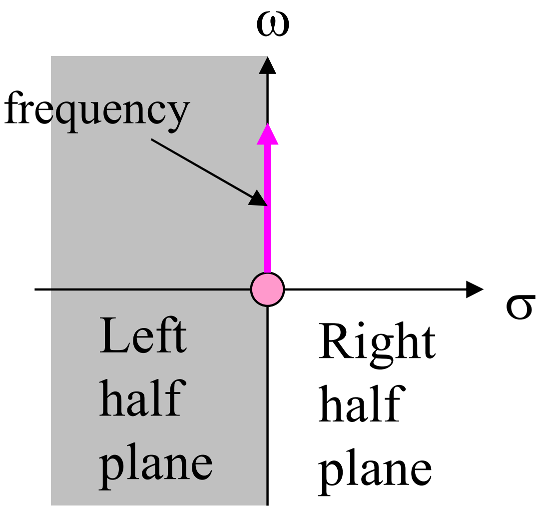
Z transform using delay z^{-n}

A discrete-time signal can be easily transformed by using Eq.(5).
You do not need to calculate by Laplace transform.

s-plane and z-plane

s-plane	Time domain	z-plane
$X_d(s) = \sum_n x(nT_s)e^{-s(nT_s)}$	$\leftarrow x_d(t) = \sum_{n=0}^{\infty} x(t)\delta(t - nT_s) \rightarrow$	$X_d(z) = \sum_{n=0}^{\infty} x_n z^{-n}$

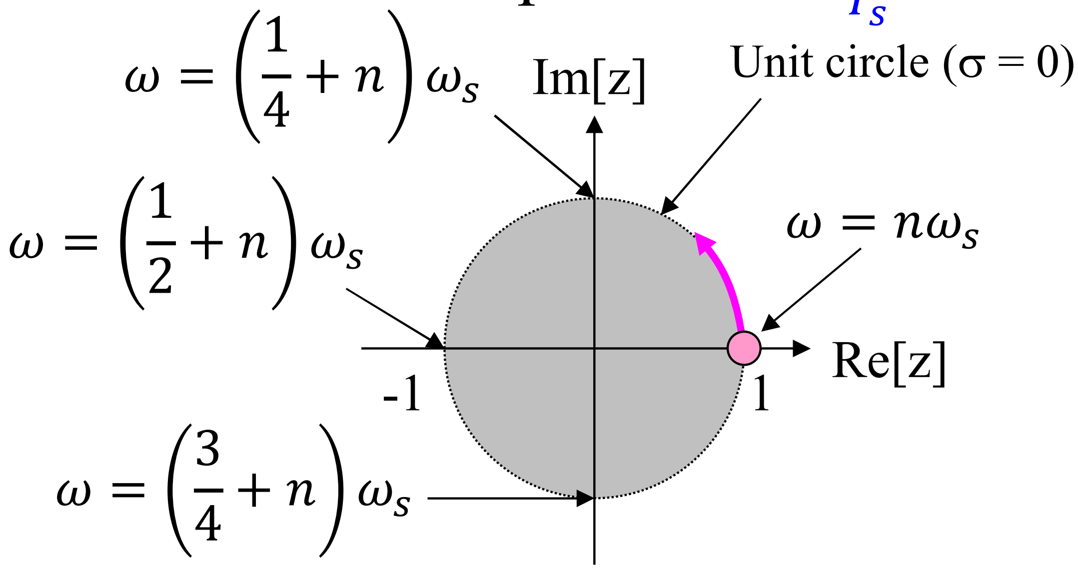
s-plane



$$s = \sigma + j\omega$$

z-plane

$$\omega_s = \frac{2\pi}{T_s}$$

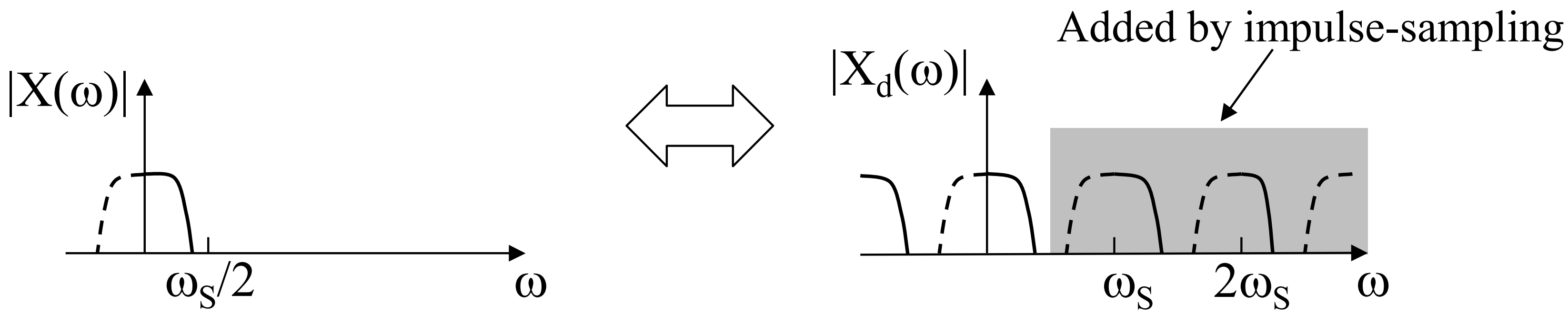


$$z = e^{sT_s} = e^{\sigma T_s}e^{j\omega T_s}$$
$$= e^{\sigma T_s}(\cos \omega T_s + j \sin \omega T_s)$$
$$= e^{\sigma T_s}(\cos \omega_s + j \sin \omega_s)$$

Spectrum of PAM signal

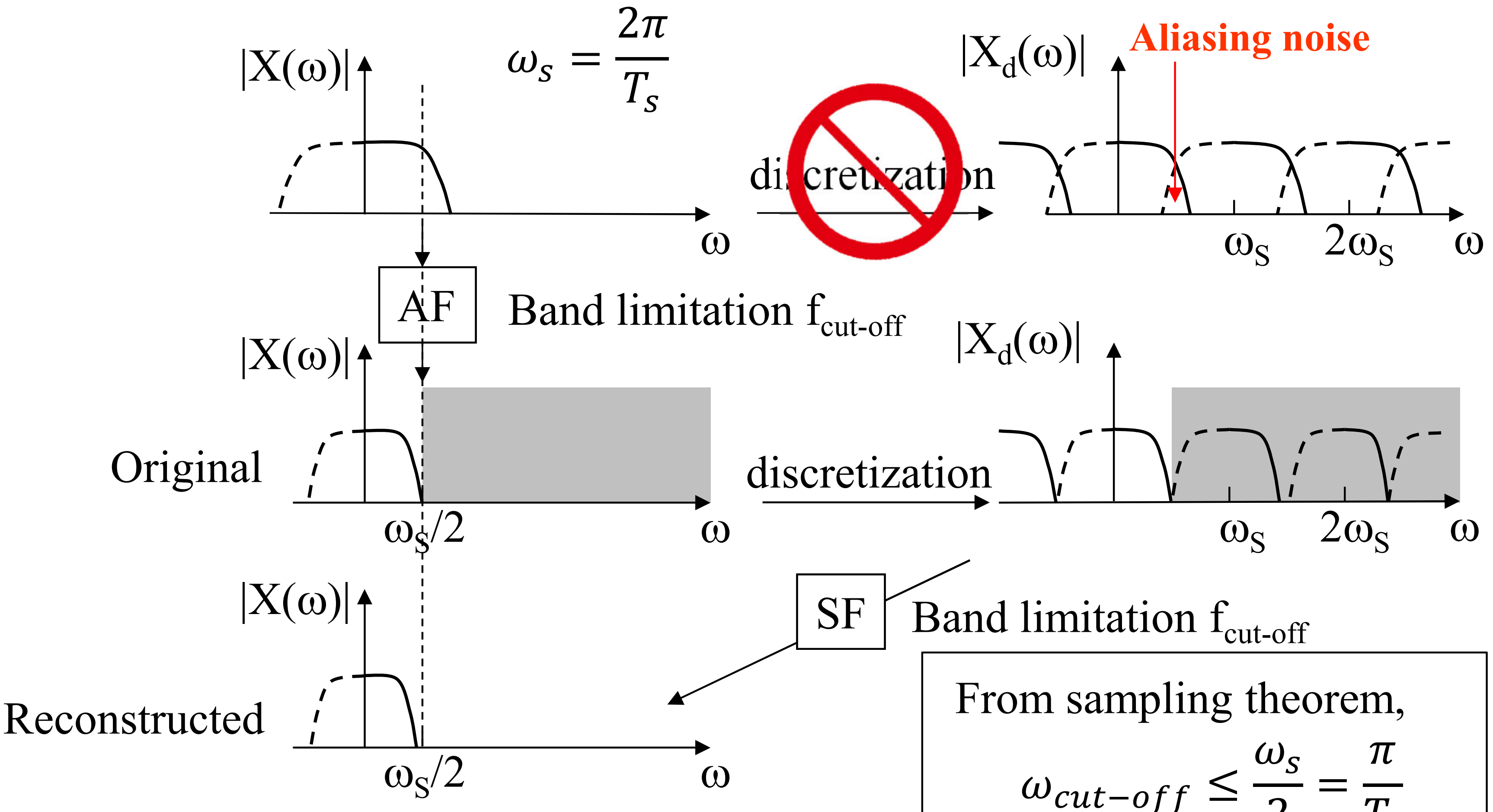
Spectrum of continuous-time signal

Spectrum of discrete-time signal



See Appendix 2(1) and 2(2).

Antialiasing and smoothing of signals

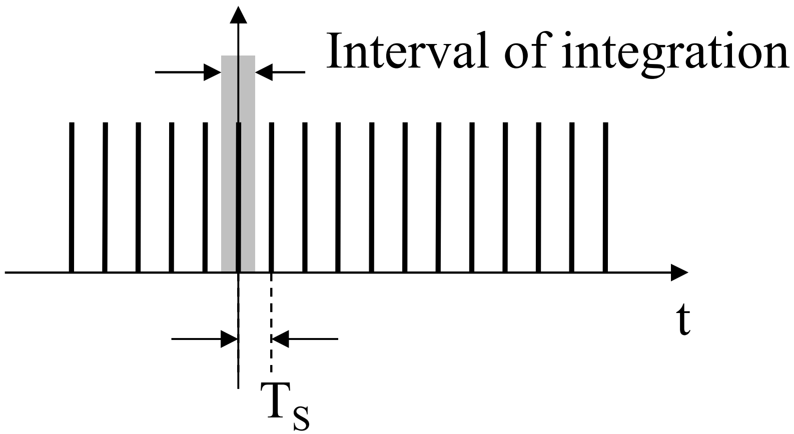


From sampling theorem,

$$\omega_{cut-off} \leq \frac{\omega_s}{2} = \frac{\pi}{T_s}$$

NOTE: AF and SF can be implemented in continuous-time circuits.

Appendix 2(1) Fourier series of impulse sampler



$$\delta_T(t) = \sum_n \delta(t - nT_S)$$

$$\delta_T(t) = \sum_n c_n e^{jn\frac{2\pi}{T_S}t}$$

$$c_n = \frac{1}{T_S}\int_{-\frac{T_S}{2}}^{\frac{T_S}{2}} \delta_T(t)\cdot e^{jn\frac{2\pi}{T_S}t}\,dt$$

$$= \frac{1}{T_S} e^{jn\frac{2\pi}{T_S}0} = \frac{1}{T_S}$$

$$\therefore \delta_T(t) = \frac{1}{T_S}\sum_n e^{jn\frac{2\pi}{T_S}t}$$

Appendix 2(2) Spectrum of PAM

$$x_d(t) = \sum_n x(t) \cdot \delta(t - nT_S) = x(t) \cdot \delta_T(t) = \frac{1}{T_S} \sum_n x(t) \cdot e^{jn\frac{2\pi}{T_S}t}$$

$\downarrow$ Translation Theorem $\mathcal{L}[e^{-at} f(t)] = F(s+a)$

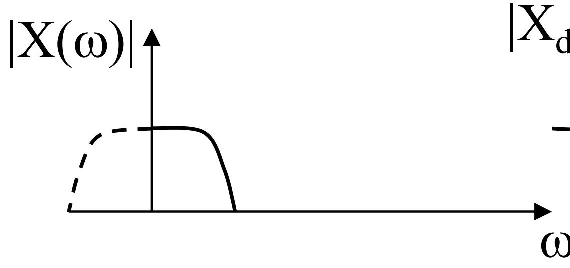
$$X_d(s) = \frac{1}{T_S} \sum_n X(s - jn\frac{2\pi}{T_S}) = \frac{1}{T_S} \sum_n X(s - jn\omega_S)$$

$$\omega_S = \frac{2\pi}{T_S}$$

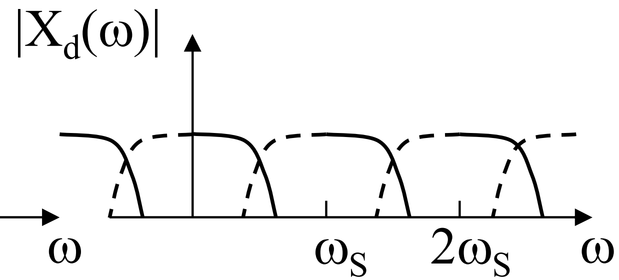
Let's say, $s = j\omega$

$$X_d(\omega) = \frac{1}{T_S} \sum_n X\{j(\omega - n\omega_S)\}$$

Continuous-time signal



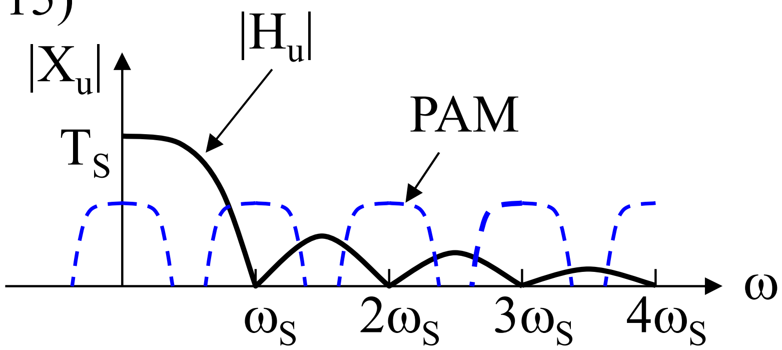
PAM signal



Appendix 2(3) Spectrum of zero-order hold

Laplace transform of zero-order hold signal (Slide 15)

$$X_u(s) = \underbrace{\boxed{\frac{1 - s^{-sT_s}}{s}}}_{\text{Transfer function of step sampler } H_u(s)} \underbrace{\sum_n x(nT_s) z^{-n}}_{\text{PAM signal}}$$



$$H_u(s) = \frac{1 - e^{-sT_S}}{s} = e^{-\frac{sT_S}{2}} \frac{e^{\frac{sT_S}{2}} - e^{-\frac{sT_S}{2}}}{s} \xrightarrow{s=j\omega} e^{-j\frac{\omega T_S}{2}} \frac{e^{j\frac{\omega T_S}{2}} - e^{-j\frac{\omega T_S}{2}}}{j\omega} = e^{-j\frac{\omega T_S}{2}} T_S \frac{\sin \frac{\omega T_S}{2}}{\frac{\omega T_S}{2}}$$

$$= \underbrace{e^{-j\frac{\omega T_S}{2}}}_{\text{Phase}} \underbrace{T_S \mathrm{Sinc}(\frac{\omega T_S}{2})}_{\text{Amplitude}}$$

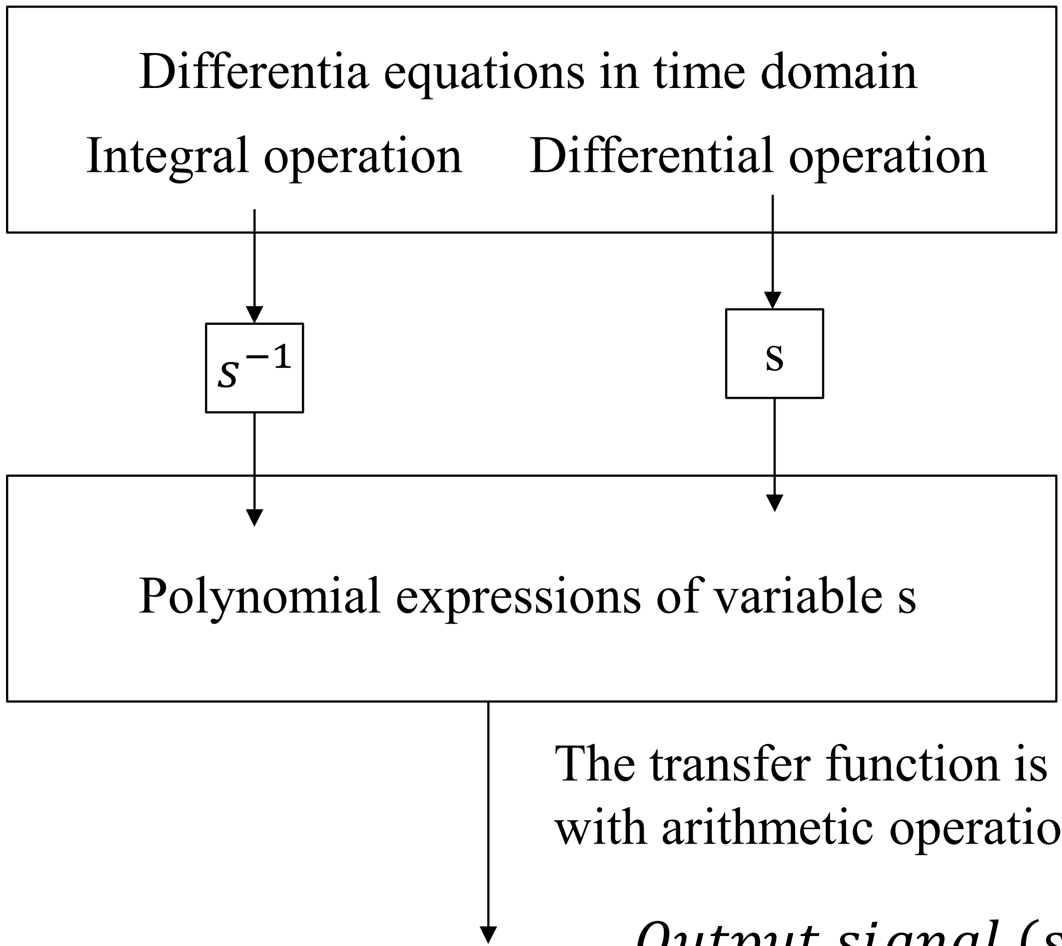
NOTE: The spectrum of zero-order hold signal is deviated from the spectrum of PAM signal by the step sampler $H_u(s)$. Therefore, the smoothing filter after DAC must has the Sinc^{-1} characteristic.

2.3 Transfer function of continuous-time analog circuits

Integration and Differentiation

Linear circuit equation in time domain

Linear circuit equation of s-variable



The transfer function is solvable with arithmetic operations .

Transfer function $H(s) = \dfrac{Output\ signal\ (s)}{Input\ signal\ (s)}$

Definition of transfer function

- Transfer function
  - $s = \sigma + j\omega$: Transfer function
  - $s = j\omega$: Frequency domain transfer function

$$H(s) = \frac{Output\ signal\ (s)}{Input\ signal\ (s)}$$

- Pole and Zero
  - $1/H(s_p) = 0$, for the complex number s_p at a location of pole in s-plane
  - $H(s_z) = 0$, for the complex number s_z at a location of zero in s-plane

- Corner frequency of pole and zero
  - A corner frequency in Bode diagram is observed as a consequence of pole and zero.
  - Pole frequency: The corner of amplitude response is convex downward.
  - Zero frequency: The corner of amplitude response is convex upward.

Decibel (dB)

The vertical axis of Bode diagrams is plotted in the decibel scale (dB).The decibel indicates the absolute value ratio of the signal amplitude.

Decibel of voltage and current signal $dB = 20log_{10}\left|\frac{V2}{V1}\right| = 20log_{10}|H(\omega)|$

Decibel of signal power $dB = 10log_{10}\left|\frac{P2}{P1}\right|$

Note: dBm is not ratio, but the absolute value of the signal power in mW.

$$dBm = 10log_{10}P(mW)$$

1-pole transfer function

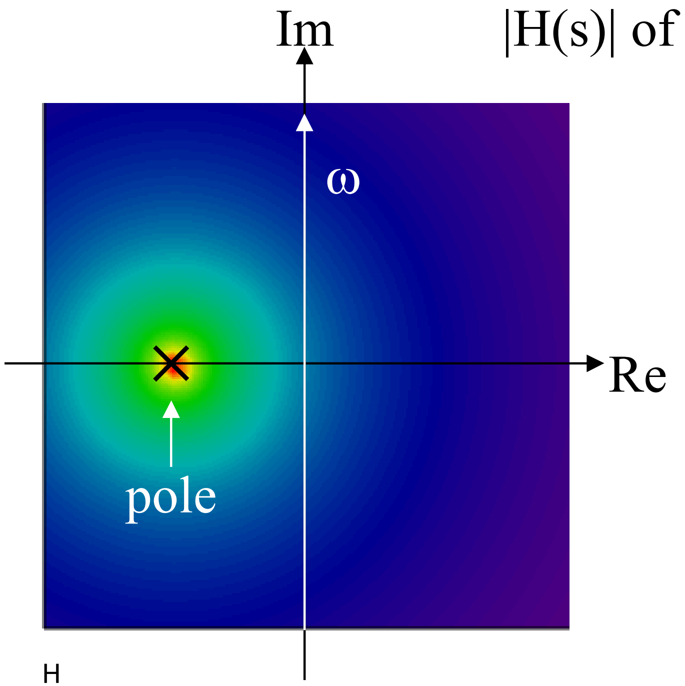
$$H(s)=\frac{a\cdot s+b}{s+c}=\frac{b}{c}\frac{(1+\frac{as}{b})}{(1+\frac{s}{c})}$$

(a, b, c = real number)

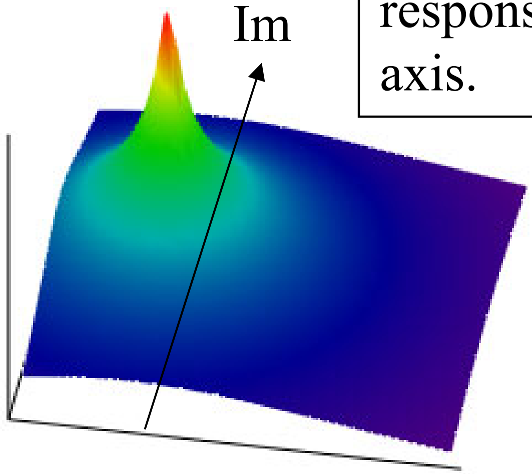
Type of frequency response

$a=0,\quad b\neq 0$	LPF
$a\neq 0,\quad b=0$	HPF

|H(s)| of LPF



$$H(s)=\frac{b}{s+c}$$



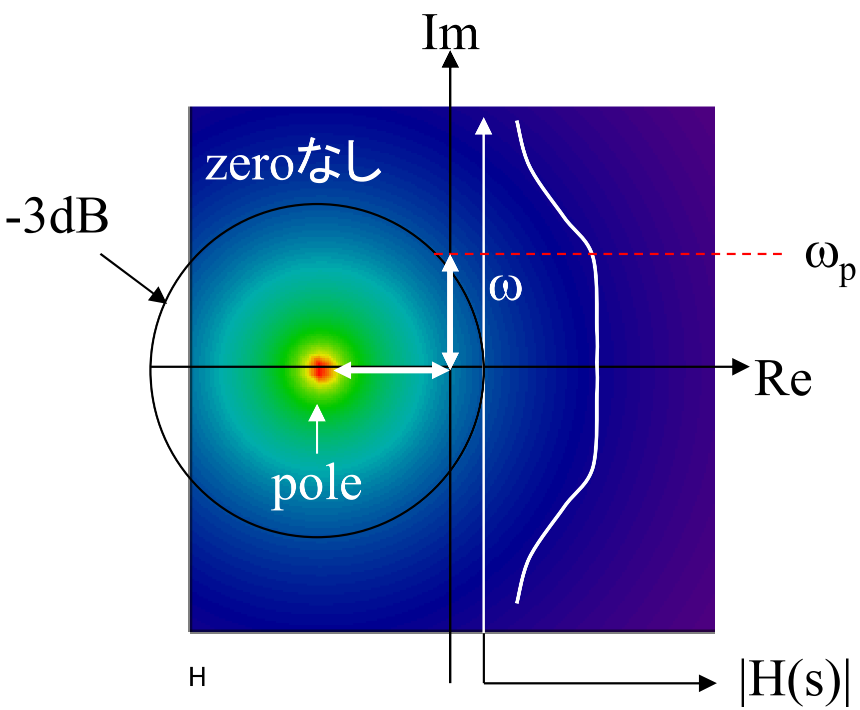
The Bode diagram is represents the amplitude response in the imaginary axis.

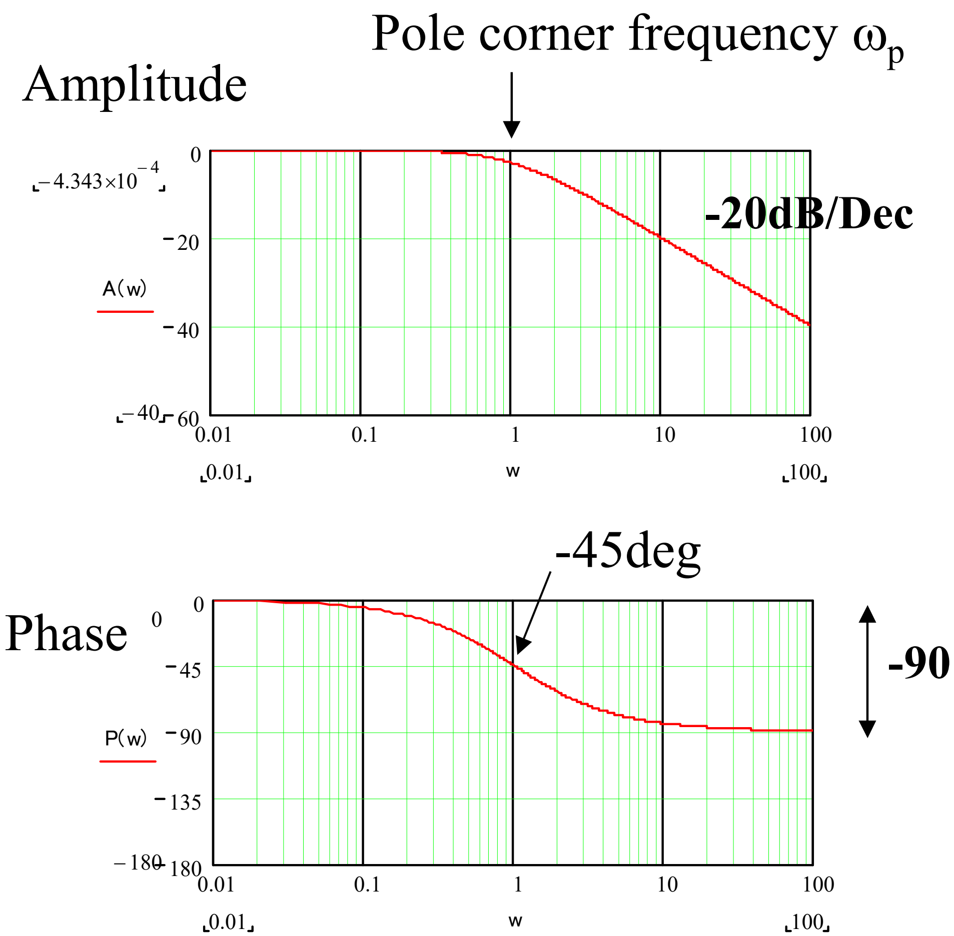
H

Bode diagram of 1st order LPF

Transfer function

$$H(s) = \frac{b}{s + c}$$



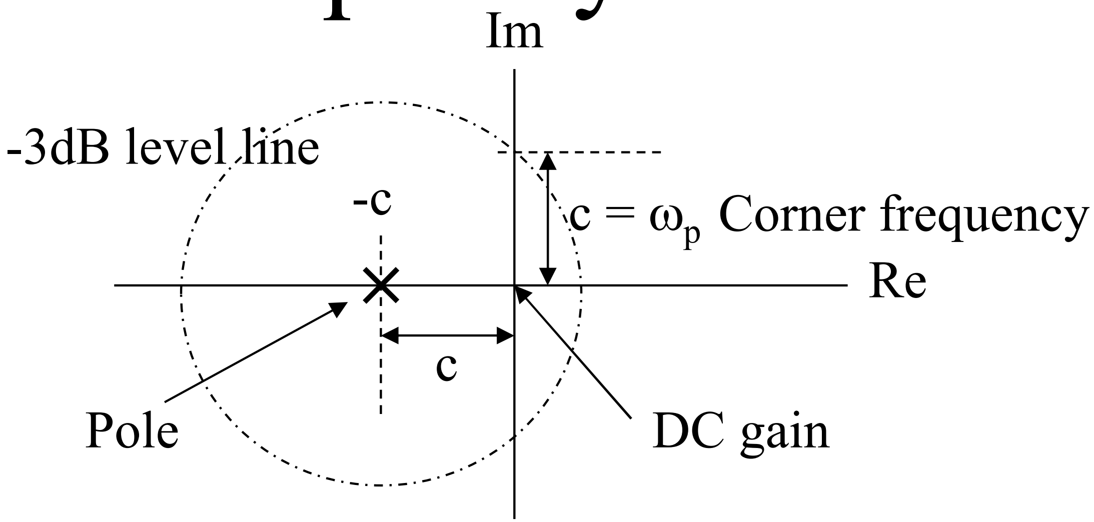


Positional relation between pole and corner frequency

$$H(s)=\frac{b}{s+c}$$

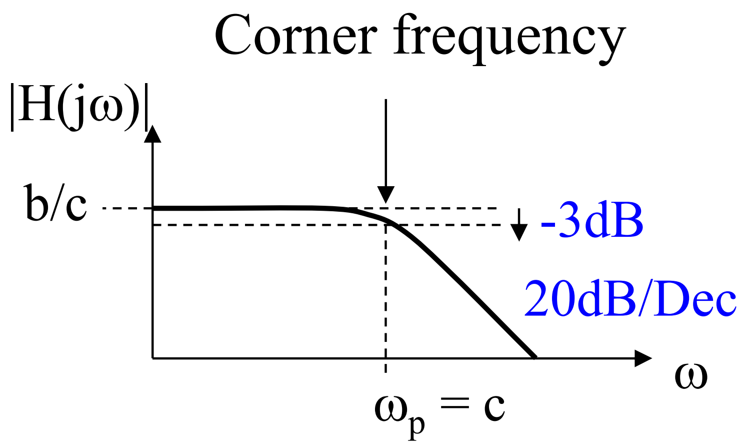
$$H(j\omega)=\frac{\frac{b}{c}}{1+j\frac{\omega}{c}}$$

$$|H(j\omega)|=\frac{\frac{b}{c}}{\sqrt{1+\frac{\omega^2}{c^2}}}$$



$$\begin{cases} \omega << c \rightarrow |H(j\omega)|=\dfrac{b}{c} \\ \omega = c \rightarrow |H(j\omega)|=\dfrac{1}{\sqrt{2}}\dfrac{b}{c} \\ \omega >> c \rightarrow |H(j\omega)|=\dfrac{b}{\omega} \end{cases}$$

$$\frac{1}{\sqrt{2}}=-3\mathrm{dB}$$



1-pole, 1-zero transfer function

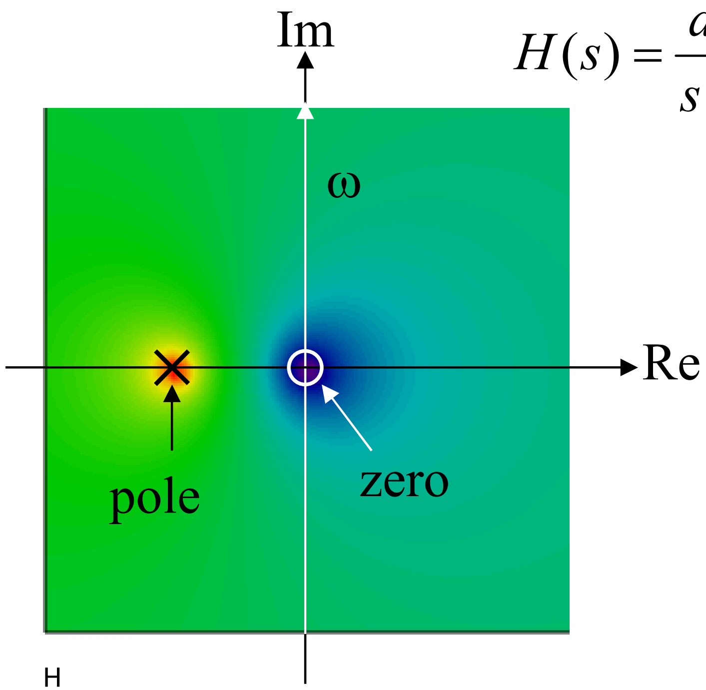
$$H(s) = \frac{a \cdot s + b}{s + c}$$ (a, b, c = real number)

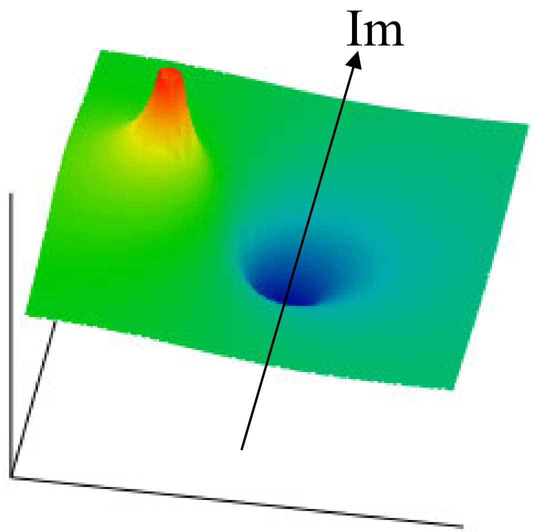
Type of frequency response

$a = 0, \quad b \neq 0$	LPF
$a \neq 0, \quad b = 0$	HPF

|H(s)| of HPF

$$H(s) = \frac{a \cdot s}{s + c}$$

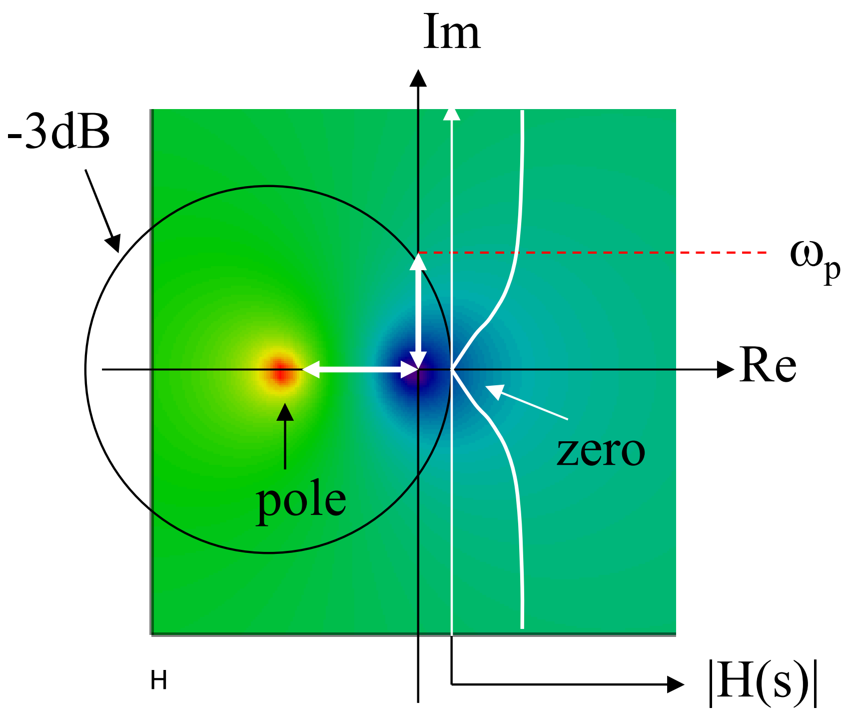


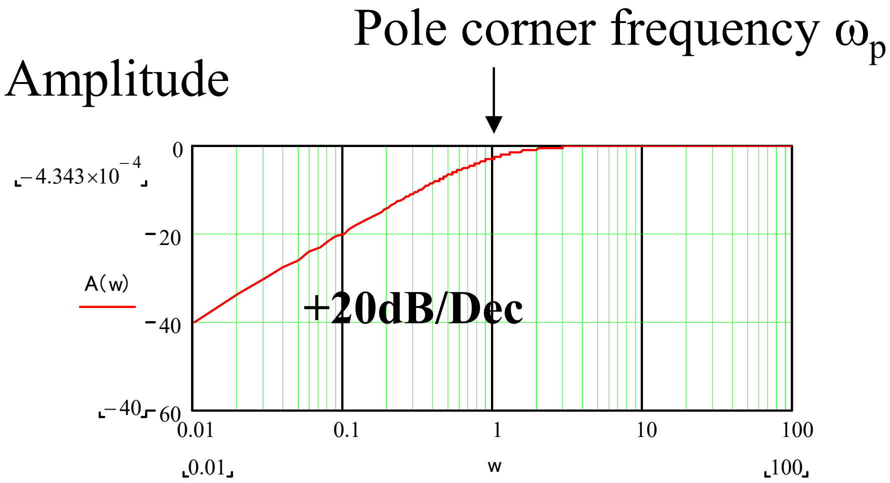


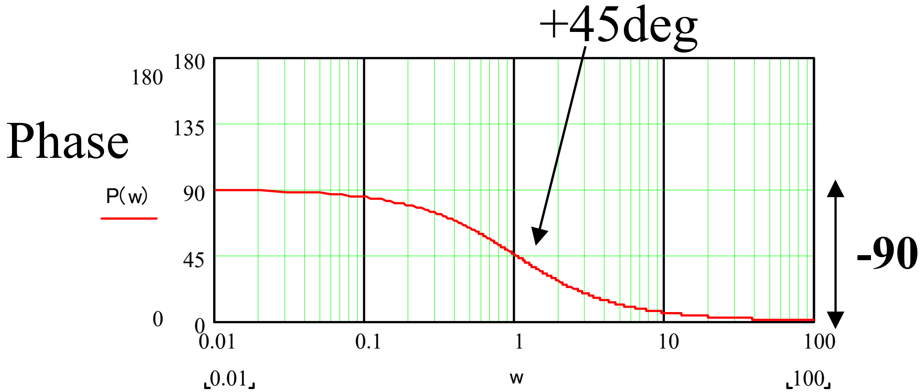
Bode diagram of 1st order HPF

Transfer function

$$H(s) = \frac{a \cdot s}{s + c}$$





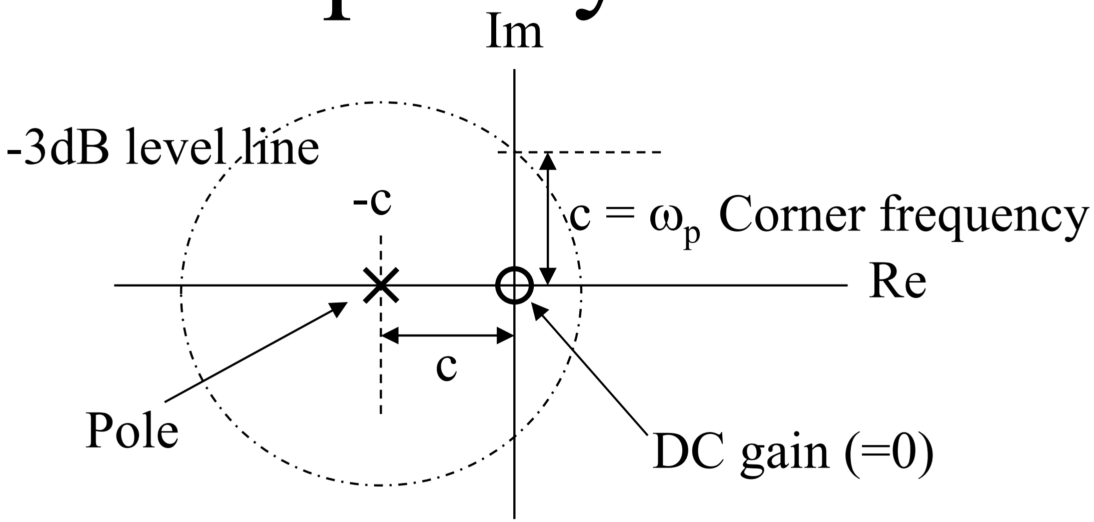


Positional relation between pole and corner frequency

$$H(s) = \frac{as}{s+c}$$

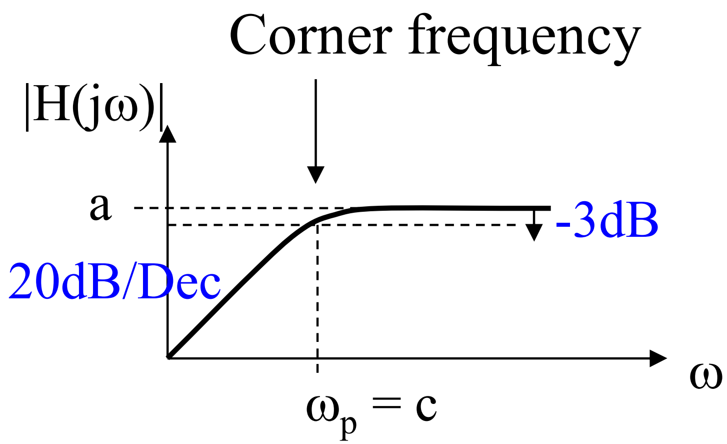
$$H(j\omega) = \frac{j\omega \dfrac{a}{c}}{1 + j\dfrac{\omega}{c}}$$

$$|H(j\omega)| = \frac{\omega \dfrac{a}{c}}{\sqrt{1 + \dfrac{\omega^2}{c^2}}}$$



$$\begin{cases} \omega << c \rightarrow |H(j\omega)| = \omega \dfrac{a}{c} \\ \omega = c \rightarrow |H(j\omega)| = \dfrac{a}{\sqrt{2}} \\ \omega >> c \rightarrow |H(j\omega)| = a \end{cases}$$

$$\frac{1}{\sqrt{2}} = -3\text{dB}$$



2-pole transfer function

$$H(s) = \frac{a \cdot s^2 + b \cdot s + c}{s^2 + d \cdot s + e}$$

(a, b, c = real number)

Type of frequency response

$a = b = 0, \quad c \neq 0$	LPF
$a = c = 0, \quad b \neq 0$	BPF
$b = c = 0, \quad a \neq 0$	HPF
$b = 0, \quad a \neq 0, \quad c \neq 0$	BEF

|H(s)| of LPF

$$H(s) = \frac{c}{s^2 + d \cdot s + e}$$

Note: If the denominator is factorable, the transfer function has 2 real poles.

Solution of the pole

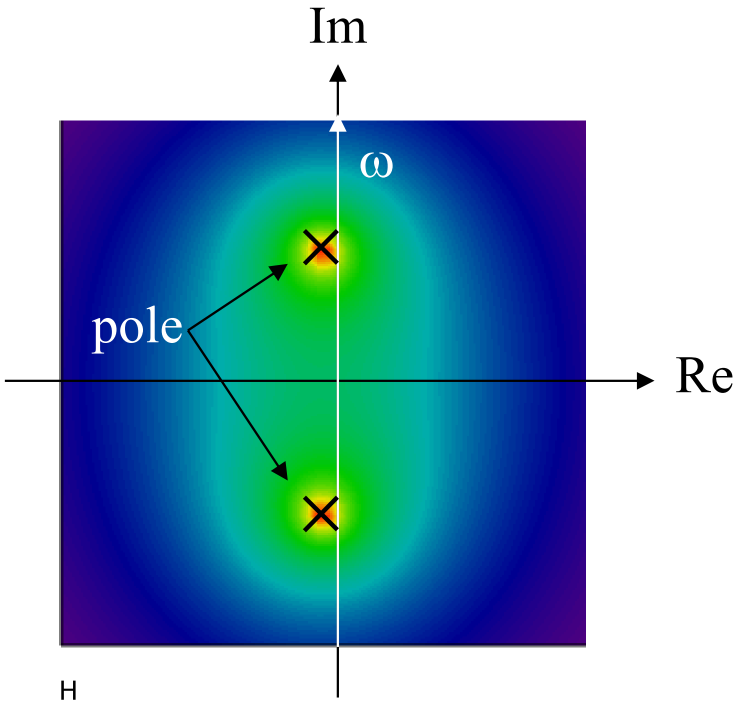
$$D(s) = s^2 + d \cdot s + e = 0$$

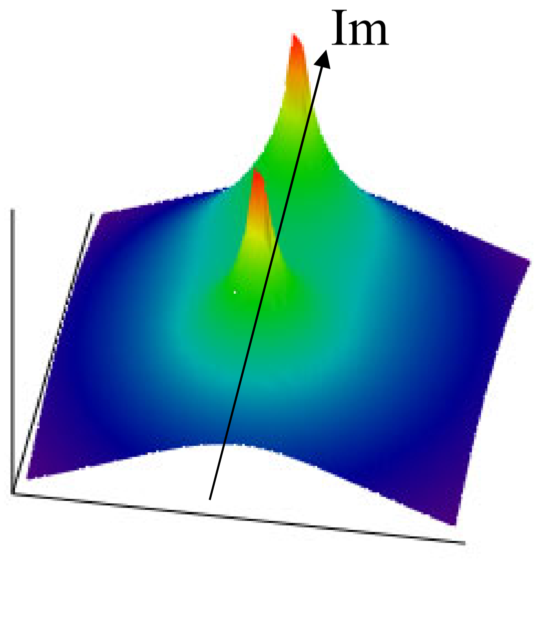
$$s = -\frac{d}{2} \pm j\sqrt{e - \frac{d^2}{4}}$$

Complex number of 2 poles

Bode diagram of 2nd order LPF

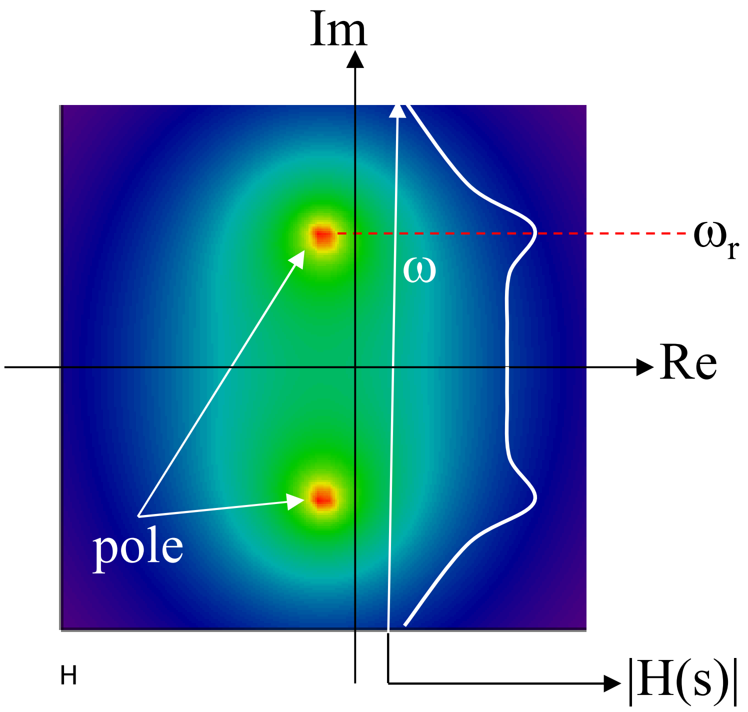
$$H(s) = \frac{c}{s^2 + d \cdot s + e}$$ の場合

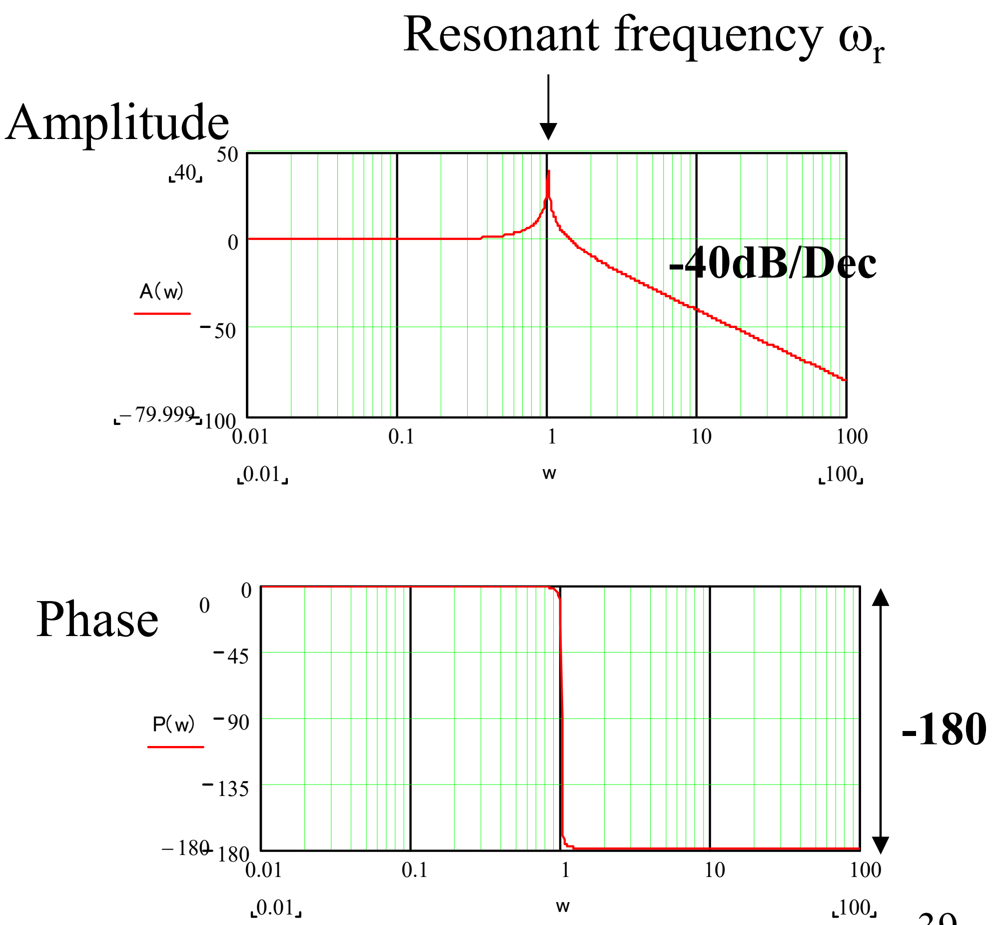




Bode diagram of 2nd order LPF

$$H(s) = \frac{c}{s^2 + d \cdot s + e}$$



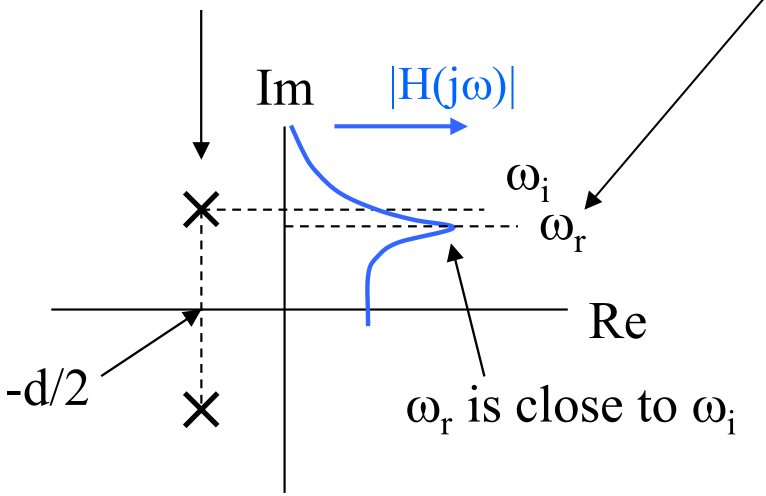


Positional relation between pole and corner frequency

$$H(s)=\frac{c}{s^2+d\cdot s+e}$$

$s^2+d\cdot s+e=0$ のとき

$$s=\underbrace{-\frac{d}{2}\pm j\sqrt{e-\frac{d^2}{4}}}_{\text{Location of the pole in s-plane}}\equiv-\frac{d}{2}\pm j\omega_i$$



$$|H(j\omega)|=\frac{c}{\sqrt{\{(j\omega)^2+d(j\omega)+e\}\{(-j\omega)^2+d(-j\omega)+e\}}}$$

$$=\frac{c}{\sqrt{\{\omega^2-(e-\frac{d^2}{2})\}^2+d^2(e-\frac{d^2}{4})}}$$

$$\omega_r\equiv\sqrt{(e-\frac{d^2}{2})}$$

$$|H(j\omega)|=\frac{c}{\sqrt{\{\omega^2-\omega_r^2\}^2+d^2(e-\frac{d^2}{4})}}$$

If $\omega=\omega_r$, the amplitude reach a maximum.

$$|H(j\omega_r)|=\frac{c}{d\sqrt{e-\frac{d^2}{4}}}$$

The smaller d causes the higher peak.

2.4 Transfer function of discrete-time analog/digital circuits

Definition of transfer function on z-plane

$$H(s) = \frac{Output\ signal\ (s)}{Input\ signal\ (s)}$$

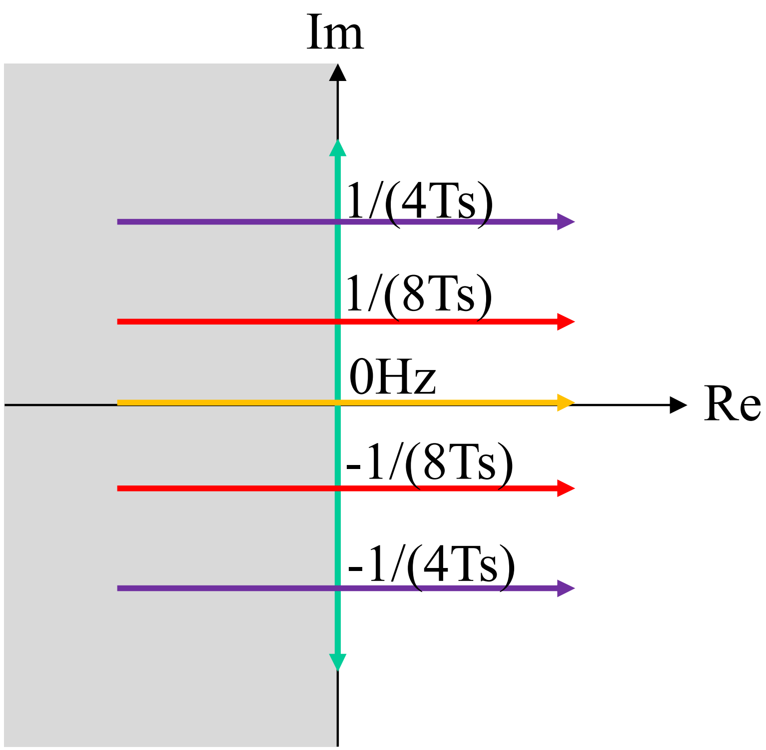
$$z^{-1} = e^{-sT_s} \quad \downarrow$$

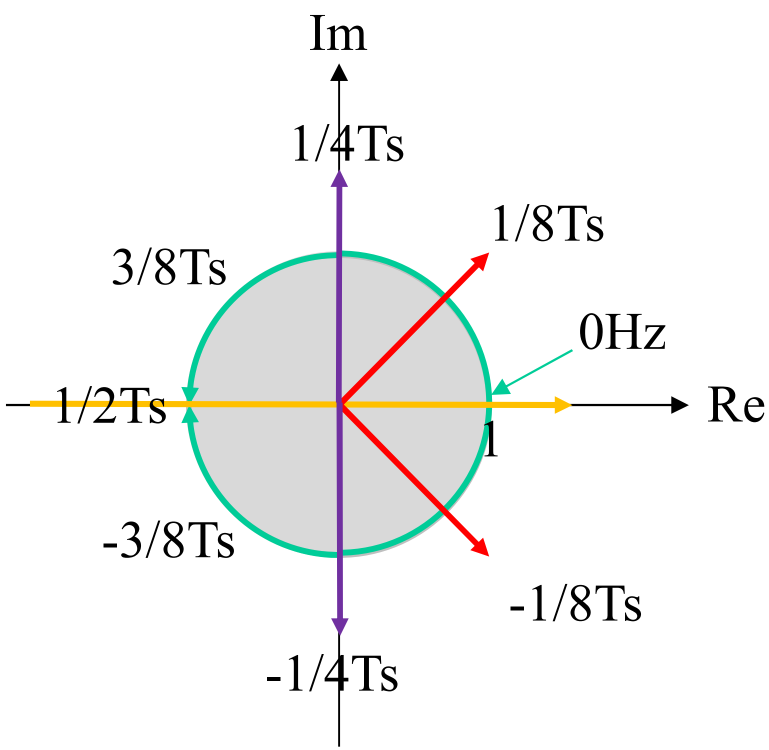
$$H(z) = \frac{Output\ signal\ (z)}{Input\ signal\ (z)}$$

The non-linear functions of z-variable are deduced from the rational functions including an integration and a differentiation of s-variable. The complexity of the circuit implementation is avoided by using a translation theorem and some approximations.

Correspondence relation of s-domain and z-domain

$$z = e^{(\sigma + j\omega)T_s}$$





Time shift in Laplace transform

$k(t) = g(t-T)u(t-T)$, where *u(t)* is a unit step function

$$\Big\downarrow \mathcal{L}$$

$$K(s) = \int_0^\infty g(t-T)u(t-T)e^{-st}dt$$

$$\tau = t - T$$

$$K(s) = \int_0^\infty g(\tau)u(\tau)e^{-s(\tau+T)}d(\tau+T)$$

$$= \int_0^\infty g(\tau)u(\tau)e^{-s(\tau+T)}d\tau = e^{-sT}\int_0^\infty g(\tau)u(\tau)e^{-s\tau}d\tau$$

$$= e^{-sT}\int_0^\infty f(\tau)e^{-s\tau}d\tau = e^{-sT}F(s) \quad \longleftarrow \quad f(t) = g(t)u(t)$$

$\Big\downarrow \mathcal{Z}$ if $T = nT_s$ (T_s: Sampling period)

$$K(z) = e^{-snT_s}F(z) = z^{-n}F(z)$$

Time shift in Z transform

$$x(nT_s) \xrightarrow{z} X(z) \qquad \text{, where } t = nT_s \tag{6}$$

(Right shift)
$$x(nT_s - nT_s) \xrightarrow{z} z^{-n} X(z) \tag{7}$$

(Left shift)
$$x(nT_s + mT_s) \xrightarrow{z} z^{m} X(z) - z^{m} \sum_{k=0}^{m-1} x(kT_s) z^{-k} \tag{8}$$

$$x_{in}(t) \longrightarrow \boxed{\text{Delay } T_s} \longrightarrow x_{out}(t) \qquad X_{in}(z) \longrightarrow \boxed{z^{-1}} \longrightarrow X_{out}(z)$$

The equation 7 shows that the multiplication of z^{-n} in z-plane is equivalent to the delay of nT_s in time domain. The z^{-1} operator is called "Delay element".

Approximation of Z transform 1

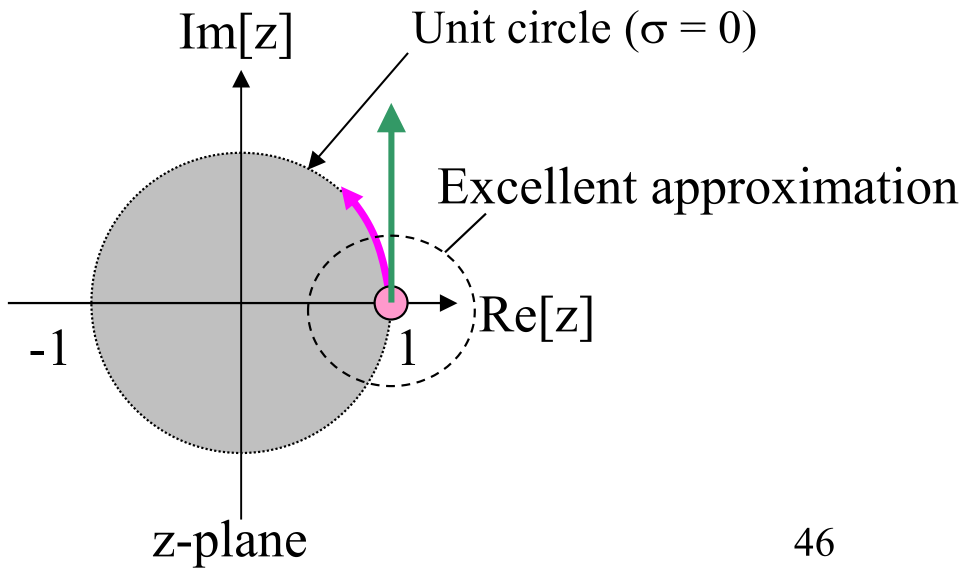
- Forward Euler Transformation (FET)

Power series expansion of z at s = 0

$$z = e^{sT_S} = 1 + \frac{1}{1!}(sT_S)^1 + \frac{1}{2!}(sT_S)^2 + \frac{1}{3!}(sT_S)^3 + \Lambda$$

$$\approx 1 + sT_S$$

$$\therefore s \approx \frac{z-1}{T_S}$$



Approximation of Z transform 2

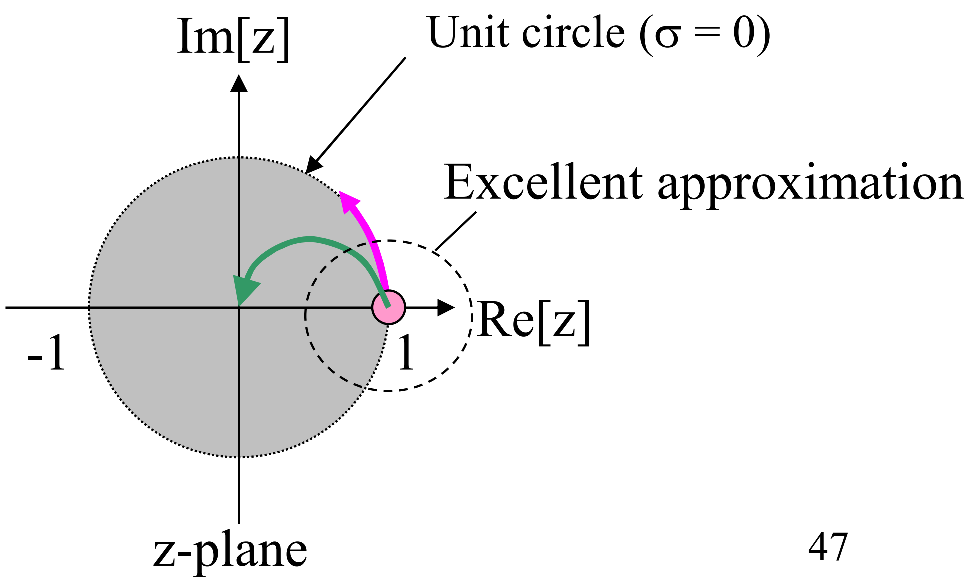
- Backward Euler Transformation (BET)

Power series expansion of z^{-1} at $s = 0$

$$z^{-1} = e^{-sT_S} = 1 - \frac{1}{1!}(sT_S)^1 + \frac{1}{2!}(sT_S)^2 - \frac{1}{3!}(sT_S)^3 + \Lambda$$

$$\approx 1 - sT_S$$

$$\therefore s \approx \frac{1 - z^{-1}}{T_S}$$



Approximation of Z transform 3

- Bilinear Transformation

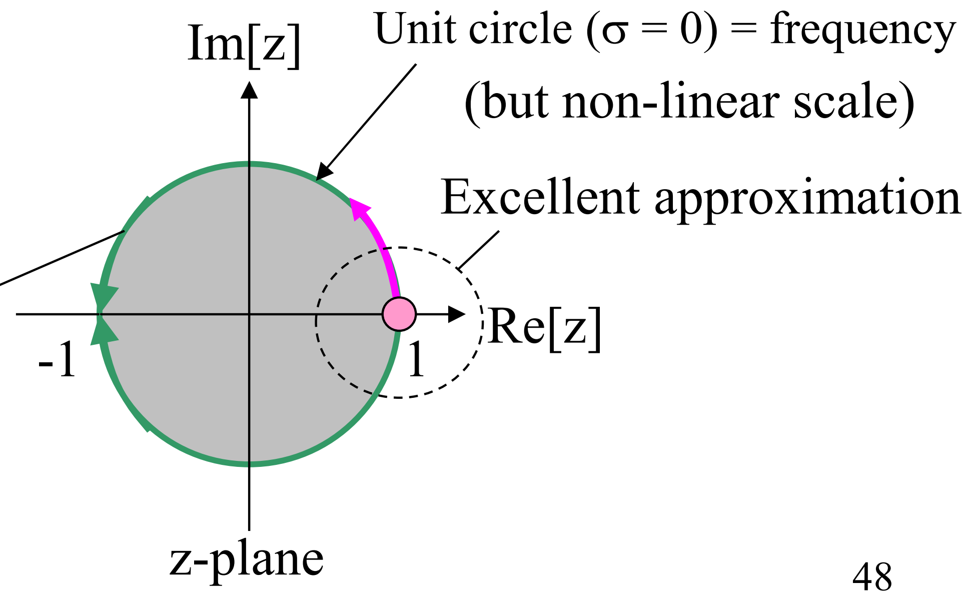
Power series expansion of $\log_e z$ at s = 0

$$sT_S = \ln z = 2 \cdot [\frac{z-1}{z+1} + \frac{1}{3}\frac{(z-1)^3}{(z+1)^3} + \frac{1}{5}\frac{(z-1)^5}{(z+1)^5} + \Lambda]$$

$$\approx 2\frac{z-1}{z+1}$$

$$\therefore s \approx \frac{2}{T_S}\frac{z-1}{z+1} = \frac{2}{T_S}\frac{1-z^{-1}}{1+z^{-1}}$$

$$\omega_{bilinear} = \frac{2}{T_s} arctan \frac{\omega T_s}{2}$$



Integration of continuous-time signal

$$v_{in}(t) \longrightarrow \boxed{\text{Integrator}} \longrightarrow v_{out}(t)$$

$$v_{out}(t) = \int_0^t v_{in}(\tau)d\tau$$

Laplace transform

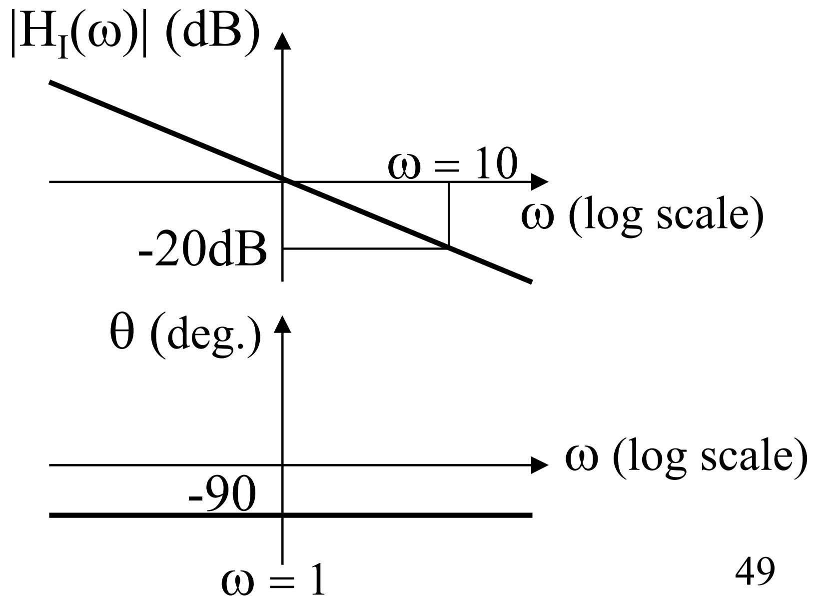
$$V_{out}(s) = \frac{1}{s}V_{in}(s) + \frac{1}{s}\int_{-\infty}^{0} v_{in}(\tau)d\tau$$
$$= \frac{1}{s}V_{in}(s)$$

(where the initial condition is ignored.)

Frequency domain transfer function ($s = j\omega$)

$$H_I(\omega) = \frac{1}{s} = \frac{1}{j\omega} = -j\frac{1}{\omega}$$

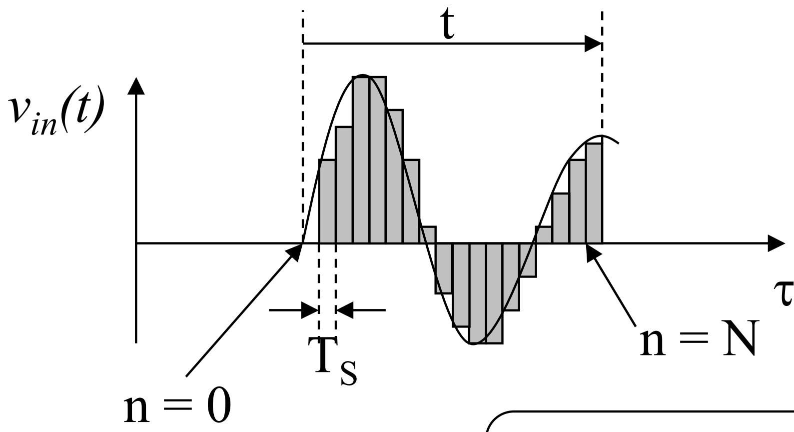
$$\left|H_I(\omega)\right|[dB] = 20\log\left|H_I(\omega)\right| = 20\log\frac{1}{\omega} = -20\log\omega$$



Integration of discrete-time signal

Integration approximated with BET

$$H_I(s) = \frac{1}{s} \cong \frac{T_S}{1 - z^{-1}}$$



$$v_{out}(t) = \int_0^t v_{in}(\tau)d\tau$$

$\Downarrow$ Discretized

$$v_{out}(t) = \sum_{n=0}^{N} v_{in}(t - nT_S)\cdot T_S \quad \xRightarrow{Z} \quad V_{out}(z) = T_S \sum_{n=0}^{N} z^{-n} V_{in}(z)$$

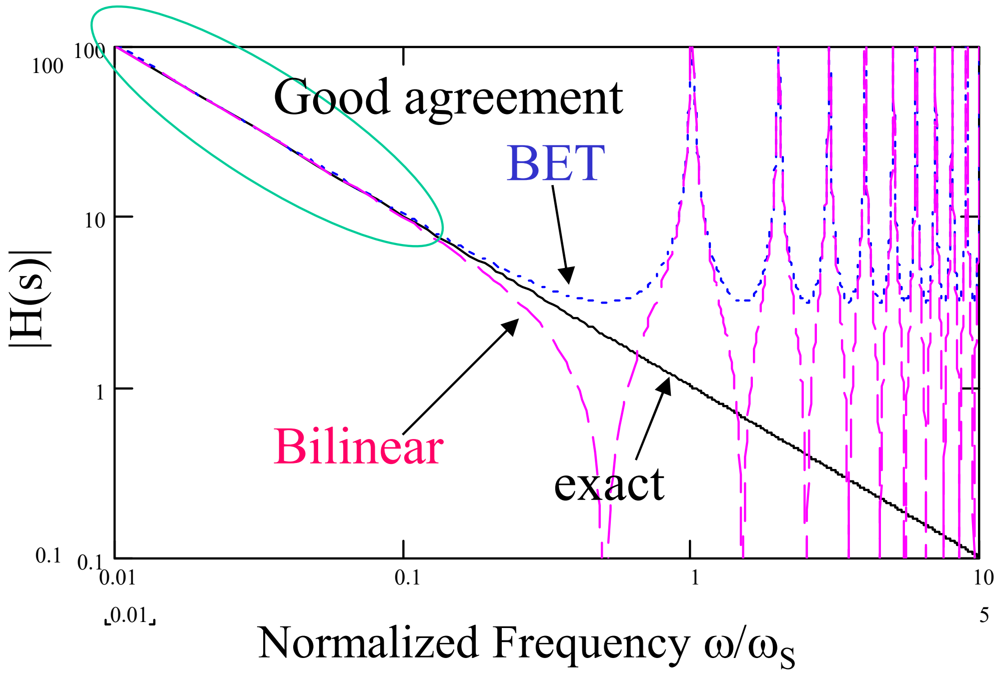
Geometrical series of z^{-1}

$$\xrightarrow{N=\infty} \underbrace{\frac{T_S}{1 - z^{-1}}}_{H_I(s)} V_{in}(z)$$

Integration error due to approximation

Frequency domain transfer function of integrator $H(s) = \frac{1}{s} = -j\frac{1}{\omega}$

$$\sigma = 0 \rightarrow \quad z^{-1} = e^{-sT_S} = e^{-j2\pi\frac{\omega}{\omega_S}}$$



BET of Integrator

$$\frac{T_S}{1-z^{-1}} = \frac{T_S}{1-e^{-j\omega T_S}} = \frac{T_S}{1-e^{-j2\pi\frac{\omega}{\omega_S}}}$$

Bilinear transformation of integrator

$$\frac{T_S}{2}\frac{1+z^{-1}}{1-z^{-1}} = \frac{T_S}{2}\frac{1+e^{-j\omega T_S}}{1-e^{-j\omega T_S}} = \frac{T_S}{2}\frac{1+e^{-j2\pi\frac{\omega}{\omega_S}}}{1-e^{-j2\pi\frac{\omega}{\omega_S}}}$$

The approximation is excellent in $\omega \ll \omega_s/2$.

Differentiation of continuous-time signal

$v_{in}(t) \longrightarrow$ differentiator $\longrightarrow v_{out}(t)$

$$v_{out}(t) = \frac{d}{dt} v_{in}(\tau)$$

$\Downarrow$ Laplace transform

$$V_{out}(s) = sV_{in}(s) - v_{in}(0)$$

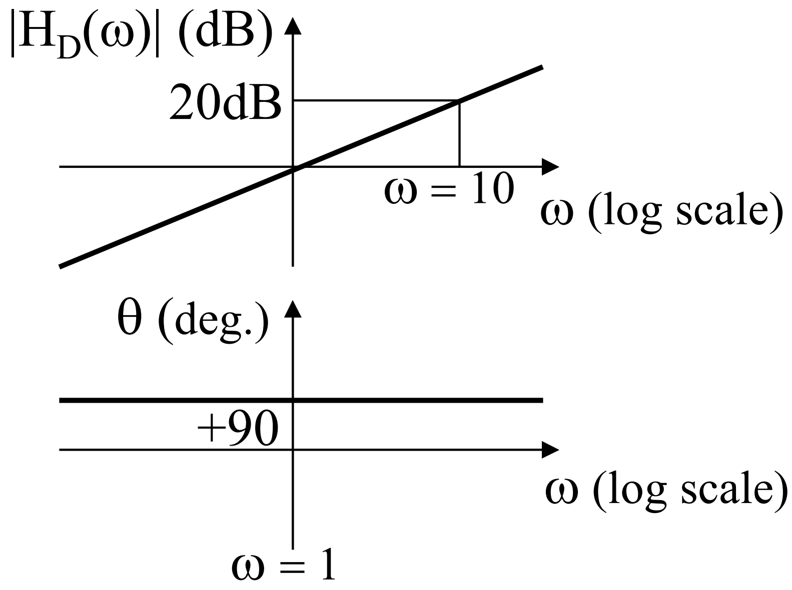
$$V_{out}(s) = sV_{in}(s)$$

(where the initial condition is ignored.)

Frequency domain transfer function (s = jω)

$$H_D(s) = s = j\omega$$

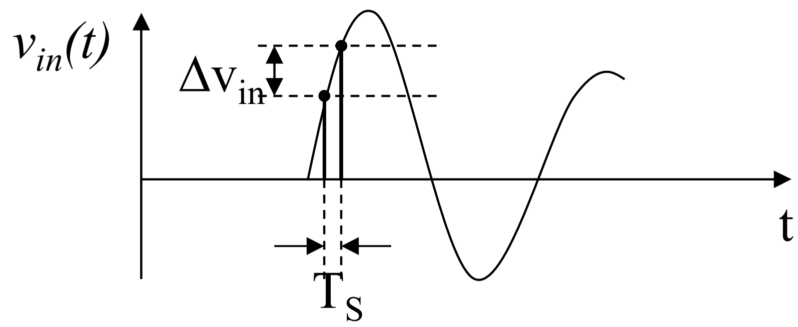
$$|H_D(\omega)|[dB] = 20\log|H_D(\omega)| = 20\log\omega$$



Differentiation of discrete-time signal

Differentiation approximated with BET

$$H_D(s) = s \cong \frac{1 - z^{-1}}{T_S}$$



$$v_{out}(t) = \frac{d}{dt} v_{in}(\tau)$$

⇓ Discretized

$$v_{out}(t) = \frac{v_{in}(t) - v_{in}(t - T_S)}{T_S}$$

$\xRightarrow{Z}$

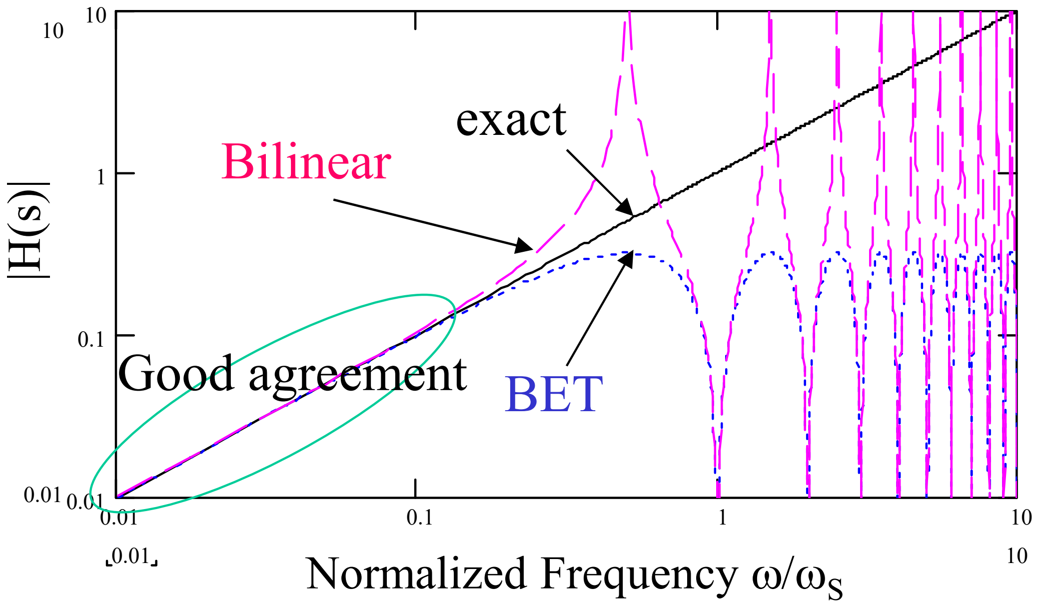
Delay element

$$V_{out}(z) = \frac{V_{in}(z) - z^{-1} V_{in}(z)}{T_S}$$

$$= \underbrace{\frac{1 - z^{-1}}{T_S}}_{H_D(s)} V_{in}(z)$$

Differentiation error due to approximation

$$\begin{cases} \text{Frequency domain transfer function of integrator} \quad H(s) = s = j\omega \\ \sigma = 0 \rightarrow \; z^{-1} = e^{-sT_S} = e^{-j2\pi\frac{\omega}{\omega_S}} \end{cases}$$



BET of differentiator

$$\frac{1-z^{-1}}{T_S} = \frac{1-e^{-j\omega T_S}}{T_S} = \frac{1-e^{-j2\pi\frac{\omega}{\omega_S}}}{T_S}$$

Bilinear transformation of differentiator

$$\frac{2}{T_S}\frac{1-z^{-1}}{1+z^{-1}} = \frac{2}{T_S}\frac{1-e^{-j\omega T_S}}{1+e^{-j\omega T_S}} = \frac{2}{T_S}\frac{1-e^{-j2\pi\frac{\omega}{\omega_S}}}{1+e^{-j2\pi\frac{\omega}{\omega_S}}}$$

The approximation is excellent in $\omega \ll \omega_s/2$.